

# MEASURE DENSITY AND EXTENSION OF BESOV AND TRIEBEL–LIZORKIN FUNCTIONS

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**ABSTRACT.** We show that a domain is an extension domain for a Hajlasz–Besov or for a Hajlasz–Triebel–Lizorkin space if and only if it satisfies a measure density condition. We use a modification of the Whitney extension where integral averages are replaced by median values, which allows us to handle also the case  $0 < p < 1$ . The necessity of the measure density condition is derived from embedding theorems; in the case of Hajlasz–Besov spaces we apply an optimal Lorentz-type Sobolev embedding theorem which we prove using a new interpolation result. This interpolation theorem says that Hajlasz–Besov spaces are intermediate spaces between  $L^p$  and Hajlasz–Sobolev spaces. Our results are proved in the setting of a metric measure space, but most of them are new even in the Euclidean setting, for instance, we obtain a characterization of extension domains for classical Besov spaces  $B_{p,q}^s$ ,  $0 < s < 1$ ,  $0 < p < \infty$ ,  $0 < q \leq \infty$ , defined via the  $L^p$ -modulus of smoothness of a function.

*Keywords:* Besov space, Triebel–Lizorkin space, extension domain, measure density, metric measure space

## 1. INTRODUCTION

The restriction and extension problems for Besov spaces and Triebel–Lizorkin spaces in the setting of the Euclidean space have been studied by several authors using different methods; see for example [31], [3], [24], [38], [33], [36], [7], [30], [39] and the references therein. In particular, it is known that if  $\Omega \subset \mathbb{R}^n$  is a Lipschitz domain or an  $(\varepsilon, \delta)$ -domain, then there is a bounded extension operator from the classical Besov space  $B_{p,q}^s(\Omega)$ , defined via the  $L^p$ -modulus of smoothness of a function, to  $B_{p,q}^s(\mathbb{R}^n)$ ,  $0 < s, p, q < \infty$ ; see [38] and [7]. The analogous extension results hold for Triebel–Lizorkin spaces; see [36], [30] and [44].

Although the class of the  $(\varepsilon, \delta)$ -domains, defined in [22], is rather wide, it does not cover all domains which admit an extension property for Besov spaces or for Triebel–Lizorkin spaces. For example, by [34, Thm 5.1], some  $d$ -thick domains in  $\mathbb{R}^n$ , measured with the  $d$ -dimensional Hausdorff content, are extension domains for certain Besov and

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Triebel–Lizorkin spaces. It is also known that the trace spaces of Besov and Triebel–Lizorkin spaces to an  $n$ -regular set  $S \subset \mathbb{R}^n$  can be intrinsically characterized [39, Thm 1.3, Thm 1.6], such  $S$  admits an extension for Besov and Triebel–Lizorkin spaces defined in terms of local polynomial approximations. See also [38] and [23] for the related results.

The connection of the  $n$ -regularity condition, or, in other words, a measure density condition, and the extension property for Sobolev spaces is studied in [39] and in [14]. By [14, Thm 5], a domain  $\Omega \subset \mathbb{R}^n$  is an extension domain for  $W^{k,p}$ ,  $1 < p < \infty$ ,  $k \in \mathbb{N}$ , if and only if  $\Omega$  satisfies the measure density condition and  $W^{k,p}(\Omega)$  coincides with the Calderon–Sobolev space defined via sharp maximal functions. For fractional Sobolev spaces  $W^{s,p}(\Omega)$ ,  $0 < s < 1$ ,  $0 < p < \infty$ , which are special cases of Besov and Triebel–Lizorkin spaces when  $p = q$ , the measure density condition characterizes extension domains by [48, Thm 1.1]. A natural question to ask is whether the same statement is true for Besov and Triebel–Lizorkin spaces within the full range of parameters  $0 < p < \infty$ ,  $0 < q \leq \infty$ . Moreover, if  $0 < s < 1$ , this question can be studied in a general setting of a metric measure space. The recent development of the theory of function spaces in metric measure spaces does not only provide a uniform approach for characterizing smoothness function spaces on topological manifolds, fractals, graphs, and Carnot–Carathéodory spaces, but at the same time it gives a new point of view to the classical Besov spaces and Triebel–Lizorkin spaces on the Euclidean space.

Among several possible definitions of Besov and Triebel–Lizorkin spaces in the metric setting, the definition recently introduced in [27] appears to be very convenient for the study of extension problems. This approach is based on Hajlasz type pointwise inequalities; it leads to the classical Besov and Triebel–Lizorkin spaces in the setting of the Euclidean space and it gives a simple way to define these spaces on a measurable subset of  $\mathbb{R}^n$  or, more generally, on a metric measure space.

**Definition 1.1.** Let  $(X, d)$  be a metric space equipped with a measure  $\mu$ . A measurable set  $S \subset X$  satisfies a *measure density condition*, if there exists a constant  $c_m > 0$  such that

$$(1.1) \quad \mu(S \cap B(x, r)) \geq c_m \mu(B(x, r))$$

for all balls  $B(x, r)$  with  $x \in S$  and  $0 < r \leq 1$ .

Note that in the literature sets satisfying condition (1.1) are sometimes called regular sets, see, for example, [40]. If the measure  $\mu$  is doubling, then the upper bound 1 for the radius  $r$  is not essential, and we can replace it by any number  $0 < R < \infty$ . Roughly speaking, the measure density condition means that the set  $S$  cannot be too thin near the boundary, in particular, by [40, Lemma 2.1], it implies that  $\mu(\bar{S} \setminus S) = 0$ . In the Euclidean space, nontrivial examples of sets satisfying the measure density condition are Cantor-like sets such as Sierpiński carpets of positive measure.

Recall that if  $\mathcal{A}$  is a quasi-Banach space of measurable functions and  $S \subset X$ , an operator  $E: \mathcal{A}(S) \rightarrow \mathcal{A}(X)$  such that  $Eu|_S = u$ , for all  $u \in \mathcal{A}(S)$ , is called an *extension operator*. A domain  $\Omega \subset X$  is an  $\mathcal{A}$ -*extension domain* if there is a bounded extension operator  $E: \mathcal{A}(\Omega) \rightarrow \mathcal{A}(X)$ .

In the metric setting, a connection between the measure density condition and the extension property for Sobolev spaces has been studied in [15] and in [40]. By [15, Thm 6], [40, Thm 1.3], the measure density condition for a set  $S$  implies the existence of a bounded, linear extension operator on the Hałasz–Sobolev space  $M^{1,p}(S)$ , for all  $1 \leq p < \infty$ . In a geodesic,  $Q$ -regular metric measure space, the measure density condition characterizes extension domains for  $M^{1,p}$ ,  $1 \leq p < \infty$ , see [15].

Our first main result is an extension theorem for Hałasz–Triebel–Lizorkin spaces  $M_{p,q}^s$  and for Hałasz–Besov spaces  $N_{p,q}^s$ , see Section 2 for the definitions.

**Theorem 1.2.** *Let  $X$  be a metric measure space with a doubling measure  $\mu$  and let  $S \subset X$  be a measurable set. If  $S$  satisfies measure density condition (1.1), then there is a bounded extension operator  $E: M_{p,q}^s(S) \rightarrow M_{p,q}^s(X)$ , for all  $0 < p < \infty$ ,  $0 < q \leq \infty$ ,  $0 < s < 1$ . The operator norm of  $E$  depends on  $c_m$ ,  $p$ ,  $q$ ,  $s$  and on the doubling constant of  $\mu$ . An analogous extension result holds for Hałasz–Besov spaces  $N_{p,q}^s$ .*

As in the corresponding results for the Hałasz–Sobolev spaces  $M^{1,p}$  in [15] and for the fractional Sobolev spaces  $W^{s,p}$ ,  $0 < s < 1$ ,  $0 < p < \infty$ , in [48], the extension is independent of the parameters of a function space. In general, our extension operator is not linear. This is due to the use of a modified Whitney type extension where integral averages are replaced by medians; similar modification was previously used in [48]. But if  $p > Q/(Q + s)$ , where  $Q$  is the doubling dimension of the space, an extension operator in Theorem 1.2 can be chosen linear by employing the classical construction with integral averages.

In the Euclidean case,  $N_{p,q}^s(\mathbb{R}^n) = B_{p,q}^s(\mathbb{R}^n)$  and  $M_{p,q}^s(\mathbb{R}^n) = F_{p,q}^s(\mathbb{R}^n)$  for all  $0 < p < \infty$ ,  $0 < q \leq \infty$ ,  $0 < s < 1$ , where  $B_{p,q}^s(\mathbb{R}^n)$  and  $F_{p,q}^s(\mathbb{R}^n)$  are Besov spaces and Triebel–Lizorkin spaces defined via an  $L^p$ -modulus of smoothness, [11]; recall that the Fourier analytic approach gives the same spaces when  $p > n/(n + s)$  in the Besov case and when  $p, q > n/(n + s)$  in the Triebel–Lizorkin case. Theorem 1.2, in particular, shows that the trace spaces of the classical Besov and Triebel–Lizorkin spaces on regular sets can be characterized in terms of pointwise inequalities. Indeed, it follows that  $B_{p,q}^s(\mathbb{R}^n)|_S = N_{p,q}^s(S)$  and  $F_{p,q}^s(\mathbb{R}^n)|_S = M_{p,q}^s(S)$  with equivalent norms.

The following statement, which is a combination of Theorem 1.2 and Theorem 6.1, is our second main result.

**Theorem 1.3.** *Let  $X$  be a  $Q$ -regular, geodesic metric measure space and let  $\Omega \subset X$  be a domain. The following conditions are equivalent:*

- (1)  $\Omega$  satisfies measure density condition (1.1);
- (2)  $\Omega$  is an  $M_{p,q}^s$ -extension domain for all  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ ;
- (3)  $\Omega$  is an  $M_{p,q}^s$ -extension domain for some values of parameters  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ .
- (4)  $\Omega$  is an  $N_{p,q}^s$ -extension domain for all  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ ;
- (5)  $\Omega$  is an  $N_{p,q}^s$ -extension domain for some values of parameters  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ .

To our knowledge, the fact that an extension domain for the Besov space, or for the Triebel–Lizorkin space, necessarily satisfies the measure density condition is also new

in the Euclidean setting; except for the special case  $p = q$  which was earlier proved in [48]. Let us also mention that the assumption on  $X$  to be geodesic is used only to guarantee the property that the boundary of each metric ball has zero measure.

In order to show that extension domains satisfy the measure density property, we need a suitable Sobolev type embedding theorem. For Hajłasz–Triebel–Lizorkin spaces such an embedding is easy to get, since they are subspaces of fractional Hajłasz–Sobolev spaces. To obtain an embedding theorem for Hajłasz–Besov spaces, we show in Theorem 4.1 that Hajłasz–Besov spaces are interpolation spaces between  $L^p$  and Hajłasz–Sobolev spaces, that is,

$$N_{p,q}^s(X) = (L^p(X), M^{1,p}(X))_{s,q},$$

for  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ . We found this result of independent interest; in case of  $p > 1$ ,  $q \geq 1$  it was earlier obtained in [9] under an additional assumption that the underlying space supports a weak  $(1, p)$ -Poincaré inequality.

We close the paper with an application of Theorem 1.3 to Besov and Triebel–Lizorkin spaces defined in the Euclidean space. In particular, we obtain the following result for the classical Besov spaces  $B_{p,q}^s$ ,  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ , defined via the  $L^p$ -modulus of smoothness.

**Theorem 1.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a domain. The following conditions are equivalent:*

- (1)  $\Omega$  satisfies measure density condition (1.1);
- (2)  $\Omega$  is a  $B_{p,q}^s$ -extension domain for all  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ ;
- (3)  $\Omega$  is a  $B_{p,q}^s$ -extension domain for some  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ ;

Extension problems are closely related to the question of intrinsic characterization of spaces of fractional order of smoothness on subsets  $S \subset \mathbb{R}^n$ . The obtained results shows that if  $S$  satisfies the measure density condition, then the space  $B_{p,q}^s(S)$ ,  $0 < s < 1$ ,  $0 < p < \infty$ ,  $0 < q \leq \infty$ , can be defined via the  $L^p$ -modulus of smoothness, via pointwise inequalities, in terms of an atomic decomposition, see [43] for the details on the last-mentioned approach; all these definitions would lead to the same space of functions which is the trace space of the classical Besov space  $B_{p,q}^s(\mathbb{R}^n)$ .

We also give an analogue of Theorem 1.4 for certain Triebel–Lizorkin spaces, see Theorem 7.8. Note that there are several approaches to define Triebel–Lizorkin spaces on domains, which, in general, give different spaces. In Theorem 7.8, we use a definition in the spirit of the classical definition of  $F_{p,q}^s(\mathbb{R}^n)$  via differences; it describes, for example, the trace space of  $F_{p,q}^s(\mathbb{R}^n)$  to a regular subset of the Euclidean space. Another version of Triebel–Lizorkin type spaces on domains was introduced in [36] and [30]; a characterization of extension domains for these spaces can be given similarly to the one for the Sobolev spaces in [14], see Theorem 7.11 and Remark 7.10.

The paper is organized as follows. In Section 2, we introduce the notation and the standard assumptions used in the paper and give the definitions of Hajłasz–Besov spaces and Hajłasz–Triebel–Lizorkin spaces. In Section 3, we present some auxiliary lemmas needed in the proof of our extension results. In Section 4, we prove interpolation and embedding theorems for Besov spaces. Section 5 is devoted to the proof of Theorem 1.2. In Section 6, we show that Hajłasz–Besov and Hajłasz–Triebel–Lizorkin

extension domains satisfy measure density property. In the last Section 7, we discuss the Euclidean case.

## 2. NOTATION AND PRELIMINARIES

We assume that  $X = (X, d, \mu)$  is a metric measure space equipped with a metric  $d$  and a Borel regular, doubling outer measure  $\mu$ , for which the measure of every ball is positive and finite. The *doubling* property means that there exists a fixed constant  $c_D > 0$ , called *the doubling constant*, such that

$$(2.1) \quad \mu(B(x, 2r)) \leq c_D \mu(B(x, r))$$

for every ball  $B(x, r) = \{y \in X : d(y, x) < r\}$ .

The doubling condition gives an upper bound for the dimension of  $X$  since it implies that there is a constant  $C = C(c_D) > 0$  such that for  $Q = \log_2 c_D$ ,

$$(2.2) \quad \frac{\mu(B(y, r))}{\mu(B(x, R))} \geq C \left( \frac{r}{R} \right)^Q$$

for every  $0 < r \leq R$  and  $y \in B(x, R)$ .

As a special case of doubling spaces we consider  $Q$ -regular spaces. The space  $X$  is  $Q$ -regular,  $Q \geq 1$ , if there is a constant  $c_Q \geq 1$  such that

$$(2.3) \quad c_Q^{-1} r^Q \leq \mu(B(x, r)) \leq c_Q r^Q$$

for each  $x \in X$ , and for all  $0 < r \leq \text{diam } X$ . Here  $\text{diam } X$  is the diameter of  $X$ . When we assume  $X$  to be doubling, then  $Q$  refers to (2.2), and if  $X$  is  $Q$ -regular, then  $Q$  comes from (2.3).

A metric space  $X$  is *geodesic* if every two points  $x, y \in X$  can be joined by a curve whose length equals  $d(x, y)$ .

By saying that a measurable function  $u: X \rightarrow [-\infty, \infty]$  is *locally integrable*, we mean that is integrable on balls. Similarly, the class of functions that belong to  $L^p(B)$ ,  $p > 0$ , in all balls  $B$ , is denoted by  $L_{\text{loc}}^p(X)$ . The *integral average* of a locally integrable function  $u$  over a measurable set  $A$  with  $0 < \mu(A) < \infty$  is

$$u_A = \int_A u \, d\mu = \frac{1}{\mu(A)} \int_A u \, d\mu.$$

The *Hardy–Littlewood maximal function* of a locally integrable function  $u$  is

$$\mathcal{M}u(x) = \sup_{0 < r < \infty} \int_{B(x, r)} |u| \, d\mu.$$

By  $\chi_E$ , we denote the characteristic function of a set  $E \subset X$ , and by  $\|u\|_\infty$ , the  $L^\infty$ -norm of  $u$ . The Lebesgue measure of a measurable set  $A \subset \mathbb{R}^n$  is denoted by  $|A|$ . In general,  $C$  is a positive constant whose value is not necessarily the same at each occurrence. When we want to stress that  $C$  depends on the other constants or parameters  $a, b, \dots$ , we write  $C = C(a, b, \dots)$ . If there is a positive constant  $C_1$  such that  $C_1^{-1}A \leq B \leq C_1A$ , we say that  $A$  and  $B$  are comparable, and write  $A \approx B$ .

**2.1. Hajlasz–Besov and Hajlasz–Triebel–Lizorkin spaces.** Besov and Triebel–Lizorkin spaces are certain generalizations of Sobolev spaces to the case of fractional order of smoothness. There are several ways to define these spaces in the Euclidean setting and spaces of Besov type and of Triebel–Lizorkin type in the setting of a metric space equipped with a doubling measure. For various definitions in a metric measure setting, see [9], [11], [16], [27], [32], [37], [47] and the references therein. In this paper, we mainly use the approach based on pointwise inequalities, introduced in [27]. An advantage of the pointwise definition is that it provides a simple way to intrinsically define function spaces on subsets.

**Definition 2.1.** Let  $S \subset X$  be a measurable set and let  $0 < s < \infty$ . A sequence of nonnegative measurable functions  $(g_k)_{k \in \mathbb{Z}}$  is a *fractional  $s$ -gradient* of a measurable function  $u: S \rightarrow [-\infty, \infty]$  in  $S$ , if there exists a set  $E \subset S$  with  $\mu(E) = 0$  such that

$$|u(x) - u(y)| \leq d(x, y)^s (g_k(x) + g_k(y))$$

for all  $k \in \mathbb{Z}$  and all  $x, y \in S \setminus E$  satisfying  $2^{-k-1} \leq d(x, y) < 2^{-k}$ . The collection of all fractional  $s$ -gradients of  $u$  is denoted by  $\mathbb{D}^s(u)$ .

For  $0 < p, q \leq \infty$  and a sequence  $\vec{f} = (f_k)_{k \in \mathbb{Z}}$  of measurable functions, we define

$$\|(f_k)_{k \in \mathbb{Z}}\|_{L^p(S, l^q)} = \|\|(f_k)_{k \in \mathbb{Z}}\|_{l^q}\|_{L^p(S)}$$

and

$$\|(f_k)_{k \in \mathbb{Z}}\|_{l^q(L^p(S))} = \|(\|f_k\|_{L^p(S)})_{k \in \mathbb{Z}}\|_{l^q},$$

where

$$\|(f_k)_{k \in \mathbb{Z}}\|_{l^q} = \begin{cases} (\sum_{k \in \mathbb{Z}} |f_k|^q)^{1/q}, & \text{when } 0 < q < \infty, \\ \sup_{k \in \mathbb{Z}} |f_k|, & \text{when } q = \infty. \end{cases}$$

**Definition 2.2.** Let  $S \subset X$  be a measurable set. Let  $0 < s < \infty$  and let  $0 < p, q \leq \infty$ . The *homogeneous Hajlasz–Triebel–Lizorkin space*  $\dot{M}_{p,q}^s(S)$  consists of measurable functions  $u: S \rightarrow [-\infty, \infty]$ , for which the (semi)norm

$$\|u\|_{\dot{M}_{p,q}^s(S)} = \inf_{\vec{g} \in \mathbb{D}^s(u)} \|\vec{g}\|_{L^p(S, l^q)}$$

is finite. The *(inhomogeneous) Hajlasz–Triebel–Lizorkin space*  $M_{p,q}^s(S)$  is  $\dot{M}_{p,q}^s(S) \cap L^p(S)$  equipped with the norm

$$\|u\|_{M_{p,q}^s(S)} = \|u\|_{L^p(S)} + \|u\|_{\dot{M}_{p,q}^s(S)}.$$

Similarly, the *homogeneous Hajlasz–Besov space*  $\dot{N}_{p,q}^s(S)$  consists of measurable functions  $u: S \rightarrow [-\infty, \infty]$ , for which

$$\|u\|_{\dot{N}_{p,q}^s(S)} = \inf_{(g_k) \in \mathbb{D}^s(u)} \|(g_k)\|_{l^q(L^p(S))}$$

is finite, and the *Hajlasz–Besov space*  $N_{p,q}^s(S)$  is  $\dot{N}_{p,q}^s(S) \cap L^p(S)$  equipped with the norm

$$\|u\|_{N_{p,q}^s(S)} = \|u\|_{L^p(S)} + \|u\|_{\dot{N}_{p,q}^s(S)}.$$



When  $0 < p < 1$ , the (semi)norms defined above are actually quasi-(semi)norms, but for simplicity we call them, as well as other quasi-seminorms in this paper, just norms.

**Remark 2.3.** Observe that for inhomogeneous Hajłasz–Triebel–Lizorkin and Hajłasz–Besov spaces the norms defined above are equivalent to

$$\|u\|_{L^p(S)} + \inf_{\vec{g} \in \mathbb{D}^s(u)} \|(g_k)_{k \in \mathbb{N}}\|_{L^p(S, l^q)} \quad \text{and} \quad \|u\|_{L^p(S)} + \inf_{\vec{g} \in \mathbb{D}^s(u)} \|(g_k)_{k \in \mathbb{N}}\|_{l^q(S, L^p)}$$

respectively, that is, it is enough to take into account only the coordinates of  $\vec{g}$  with positive indices. Indeed, if  $x, y \in S \setminus E$  and  $2^{-k-1} \leq d(x, y) < 2^{-k}$  with  $k \leq 0$ , then

$$|u(x) - u(y)| \leq |u(x)| + |u(y)| \leq 2^{(k+1)s} d(x, y)^s (|u(x)| + |u(y)|).$$

Hence, if  $(g_k)_{k \in \mathbb{Z}} \in \mathbb{D}^s(u)$ , then  $(g'_k)_{k \in \mathbb{Z}}$ , where  $g'_k = g_k$  for  $k > 0$  and  $g'_k = 2^{(k+1)s}|u|$  for  $k \leq 0$ , belongs to  $\mathbb{D}^s(u)$ . Calculating the norm, for example, for Hajłasz–Triebel–Lizorkin space, we obtain that

$$\begin{aligned} \|\vec{g}'\|_{L^p(S, l^q)} &\leq C(\|(g'_k)_{k \in \mathbb{N}}\|_{L^p(S, l^q)} + \|(g'_k)_{k \leq 0}\|_{L^p(S, l^q)}) \\ &= C\|(g_k)_{k \in \mathbb{N}}\|_{L^p(S, l^q)} + C\|u\|_{L^p(S)} \left( \sum_{k=-\infty}^0 2^{(k+1)sq} \right)^{1/q}, \end{aligned}$$

where the constants  $C$  depend on  $p$  and  $q$  only. This implies that

$$\inf_{\vec{g} \in \mathbb{D}^s(u)} \|\vec{g}\|_{L^p(S, l^q)} \leq C \left( \inf_{\vec{g} \in \mathbb{D}^s(u)} \|(g_k)_{k \in \mathbb{N}}\|_{L^p(S, l^q)} + \|u\|_{L^p(S)} \right).$$

If  $X$  supports a weak  $(1, p)$ -Poincaré inequality with  $p \in (1, \infty)$ , then for all  $q \in (0, \infty)$ , the spaces  $M_{p,q}^1(X)$  and  $N_{p,q}^1(X)$  are trivial, that is, they contain only constant functions, see [11, Thm 4.1].

The definitions formulated above are, in particular, motivated by the Hajłasz's approach to the definition of Sobolev spaces  $M^{1,p}(X)$  on a metric measure space; see [12] and [13]. The fractional spaces  $M^{s,p}(X)$  were introduced in [45], and were studied, for example, in [21] and [19].

**Definition 2.4.** Let  $S \subset X$  be a measurable set. Let  $s \geq 0$  and let  $0 < p < \infty$ . A nonnegative measurable function  $g$  is an  $s$ -gradient of a measurable function  $u$  in  $S$  if there exists a set  $E \subset S$  with  $\mu(E) = 0$  such that for all  $x, y \in S \setminus E$ ,

$$(2.4) \quad |u(x) - u(y)| \leq d(x, y)^s (g(x) + g(y)).$$

The collection of all  $s$ -gradients of  $u$  is denoted by  $\mathcal{D}^s(u)$  and the 1-gradients shortly by  $\mathcal{D}(u)$ . The *homogeneous Hajłasz space*  $\dot{M}^{s,p}(S)$  consists of measurable functions  $u$  for which

$$\|u\|_{\dot{M}^{s,p}(S)} = \inf_{g \in \mathcal{D}^s(u)} \|g\|_{L^p(S)}$$

is finite. The *Hajłasz space*  $M^{s,p}(S)$  is  $\dot{M}^{s,p}(S) \cap L^p(S)$  equipped with the norm

$$\|u\|_{M^{s,p}(S)} = \|u\|_{L^p(S)} + \|u\|_{\dot{M}^{s,p}(S)}.$$

Recall that for  $p > 1$ ,  $M^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$  [12], whereas for  $n/(n+1) < p \leq 1$ ,  $M^{1,p}(\mathbb{R}^n)$  coincides with the Hardy–Sobolev space  $H^{1,p}(\mathbb{R}^n)$  [25, Thm 1]. Notice also that  $M^{0,p}(X) = L^p(X)$  and that  $M^{s,p}(X)$  coincides with the Hajlasz–Triebel–Lizorkin space  $M_{p,\infty}^s(X)$ , see [27, Prop. 2.1] for a simple proof of this fact.

**2.2. On different definitions of Besov and Triebel–Lizorkin spaces.** In the Euclidean setting the most common ways to define Besov and Triebel–Lizorkin spaces, via the  $L^p$ -modulus of smoothness (differences) and by the Fourier analytic approach, lead to the same spaces of functions with comparable norms when  $p > n/(n+s)$  in the Besov case and when  $p, q > n/(n+s)$  in the Triebel–Lizorkin case. See, for example, [41, Chapter 2.5] and [17].

The space  $M_{p,q}^s(\mathbb{R}^n)$  given by the metric definition coincides with Triebel–Lizorkin space  $\mathbf{F}_{p,q}^s(\mathbb{R}^n)$ , defined via the Fourier analytic approach, when  $s \in (0, 1)$ ,  $p \in (n/(n+s), \infty)$  and  $q \in (n/(n+s), \infty]$ , and  $M_{p,\infty}^1(\mathbb{R}^n) = M^{1,p}(\mathbb{R}^n) = \mathbf{F}_{p,2}^1(\mathbb{R}^n)$ , when  $p \in (n/(n+1), \infty)$ . Similarly,  $N_{p,q}^s(\mathbb{R}^n)$  coincides with Besov space  $\mathbf{B}_{p,q}^s(\mathbb{R}^n)$  for  $s \in (0, 1)$ ,  $p \in (n/(n+s), \infty)$  and  $q \in (0, \infty]$ , see [27, Thm 1.2 and Remark 3.3]. For the definitions of  $\mathbf{F}_{p,q}^s(\mathbb{R}^n)$  and  $\mathbf{B}_{p,q}^s(\mathbb{R}^n)$ , we refer to [41], [42], [27, Section 3].

**2.3. Modulus of smoothness and Besov spaces.** In addition to the definition based on pointwise inequalities, we will sometimes use a generalization to the metric setting of the classical definition of the Besov spaces via the  $L^p$ -modulus of smoothness; this general version was introduced in [9].

Recall that the  $L^p$ -modulus of smoothness of a function  $u \in L^p(\mathbb{R}^n)$  is

$$(2.5) \quad \omega(u, t)_p = \sup_{|h| \leq t} \|\Delta_h(u, \cdot)\|_{L^p(\mathbb{R}^n)},$$

where  $t > 0$  and  $\Delta_h(u, x) = u(x+h) - u(x)$ . For  $0 < s < \infty$  and  $0 < p, q < \infty$ , the Besov space  $B_{p,q}^s(\mathbb{R}^n)$  consists of functions  $u \in L^p(\mathbb{R}^n)$  for which

$$\|u\|_{B_{p,q}^s(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)} + \left( \int_0^1 (t^{-s} \omega(u, t)_p)^q \frac{dt}{t} \right)^{1/q}$$

is finite (with the usual modifications when  $p = \infty$  or  $q = \infty$ ). Note that the integral over the interval  $(0, 1)$  can be replaced by the integral over  $(0, \infty)$ , since  $\omega(u, t)_p \leq C\|u\|_{L^p(\mathbb{R}^n)}$ .

Following [9] and [11], we define a modulus of smoothness which does not rely on the group structure of the underlying space and which, for a function  $u \in L^p(\mathbb{R}^n)$ , is comparable with  $\omega(u, t)_p$ .

**Definition 2.5.** Let  $t > 0$ ,  $0 < s < \infty$  and  $0 < p, q < \infty$ . Let

$$(2.6) \quad E_p(u, t) = \left( \int_X \int_{B(x,t)} |u(x) - u(y)|^p d\mu(y) d\mu(x) \right)^{1/p}.$$

The *homogeneous Besov space*  $\dot{\mathcal{B}}_{p,q}^s(X)$  consists of functions  $u \in L_{\text{loc}}^p(X)$  for which

$$\|u\|_{\dot{\mathcal{B}}_{p,q}^s(X)} = \left( \int_0^\infty (t^{-s} E_p(u, t))^q \frac{dt}{t} \right)^{1/q}$$



is finite (with the usual modification when  $q = \infty$ ). The Besov space  $\mathcal{B}_{p,q}^s(X)$  is  $\dot{\mathcal{B}}_{p,q}^s(X) \cap L^p(X)$  with the norm

$$\|u\|_{\mathcal{B}_{p,q}^s(X)} = \|u\|_{L^p(X)} + \|u\|_{\dot{\mathcal{B}}_{p,q}^s(X)}.$$

By the comparability of  $\omega(u, t)_p$  and  $E_p(u, t)$ , the space  $\mathcal{B}_{p,q}^s(\mathbb{R}^n)$  coincides with the classical space  $B_{p,q}^s(\mathbb{R}^n)$ . By [11, Thm 1.2],  $\dot{N}_{p,q}^s(X) = \dot{\mathcal{B}}_{p,q}^s(X)$  for all  $0 < s < \infty$  and  $0 < p, q \leq \infty$ , and

$$(2.7) \quad \|u\|_{\dot{N}_{p,q}^s(X)} \approx \|u\|_{\dot{\mathcal{B}}_{p,q}^s(X)}.$$

As above, the integral over the interval  $(0, \infty)$  in the norm  $\|u\|_{\mathcal{B}_{p,q}^s(X)}$  can be replaced by the integral over  $(0, 1)$ .

It also follows by the results in [11] that, for  $0 < s < 1$ ,  $0 < p, q \leq \infty$ , the Hajlasz–Triebel–Lizorkin space  $M_{p,q}^s(\mathbb{R}^n)$  coincides with the classical Triebel–Lizorkin space  $F_{p,q}^s(\mathbb{R}^n)$  defined using differences. This space consists of functions  $u \in L^p(\mathbb{R}^n)$ , for which the norm

$$\|u\|_{F_{p,q}^s(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)},$$

where

$$g(x) = \left( \int_0^1 \left( \int_{B(0,t)} |u(x+h) - u(x)|^r dh \right)^{1/r} \frac{dt}{t} \right)^{1/q}$$

and  $0 < r < \min\{p, q\}$ , is finite.

### 3. LEMMAS

This section contains lemmas needed in the proofs of the extension results.

Below we will frequently use the following simple inequality, which holds whenever  $a_i \geq 0$  for all  $i \in \mathbb{Z}$  and  $0 < p \leq 1$ ,

$$(3.1) \quad \left( \sum_{i \in \mathbb{Z}} a_i \right)^p \leq \sum_{i \in \mathbb{Z}} a_i^p.$$

The first lemma is used to estimate the norms of fractional gradients.

**Lemma 3.1.** *Let  $1 < a < \infty$ ,  $0 < b < \infty$  and  $c_k \geq 0$ ,  $k \in \mathbb{Z}$ . There is a constant  $C = C(a, b)$  such that*

$$\sum_{k \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} a^{-|j-k|} c_j \right)^b \leq C \sum_{j \in \mathbb{Z}} c_j^b.$$

*Proof.* If  $b \geq 1$ , then the Hölder's inequality for series implies that

$$\left( \sum_{j \in \mathbb{Z}} a^{-|j-k|} c_j \right)^b \leq C \sum_{j \in \mathbb{Z}} a^{-|j-k|} c_j^b.$$

If  $0 < b < 1$ , then, by (3.1),

$$\left( \sum_{j \in \mathbb{Z}} a^{-|j-k|} c_j \right)^b \leq \sum_{j \in \mathbb{Z}} a^{-b|j-k|} c_j^b.$$

Thus, denoting  $\tilde{b} = \min\{b, 1\}$ , we obtain

$$\sum_{k \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} a^{-|j-k|} c_j \right)^b \leq C \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} a^{-\tilde{b}|j-k|} c_j^b \leq C \sum_{j \in \mathbb{Z}} c_j^b \sum_{k \in \mathbb{Z}} a^{-\tilde{b}|j-k|} \leq C \sum_{j \in \mathbb{Z}} c_j^b,$$

which proves the claim.  $\square$

Next we recall the Poincaré type inequalities which are valid for functions and fractional gradients, give a definition of median values and list some of their properties and obtain certain norm estimates for Lipschitz functions.

**3.1. Poincaré type inequalities.** The definition of the fractional  $s$ -gradient implies the validity of some Sobolev–Poincaré type inequalities. A similar reasoning as in the proof of [27, Lemma 2.1] in  $\mathbb{R}^n$  gives our first inequality.

**Lemma 3.2.** *Let  $0 < s < \infty$ . Let  $u$  be a locally integrable function and let  $(g_j) \in \mathbb{D}^s(u)$ . Then, for every  $x \in X$  and  $k \in \mathbb{Z}$ ,*

$$(3.2) \quad \inf_{c \in \mathbb{R}} \int_{B(x, 2^{-k})} |u - c| d\mu \leq C 2^{-ks} \sum_{j=k-3}^k \int_{B(x, 2^{-k+2})} g_j d\mu.$$

Note that we will apply Lemma 3.2 for functions in  $N_{p,q}^s(X)$  with  $sp > Q$  and with  $sp = Q$ , and for these values of parameters functions in  $N_{p,q}^s(X)$  are locally integrable.

**Lemma 3.3** ([11], Lemma 2.1). *Let  $0 < s < \infty$  and  $0 < t < Q/s$ . Then for every  $\varepsilon$  and  $\varepsilon'$  with  $0 < \varepsilon < \varepsilon' < s$ , there exists a constant  $C > 0$  such that for all measurable functions  $u$  with  $(g_j) \in \mathbb{D}^s(u)$ ,  $x \in X$  and  $k \in \mathbb{Z}$ ,*

$$(3.3) \quad \inf_{c \in \mathbb{R}} \left( \int_{B(x, 2^{-k})} |u(y) - c|^{t^*(\varepsilon)} d\mu(y) \right)^{1/t^*(\varepsilon)} \leq C 2^{-k\varepsilon'} \sum_{j \geq k-2} 2^{-j(s-\varepsilon')} \left( \int_{B(x, 2^{-k+1})} g_j^t d\mu \right)^{1/t},$$

where  $t^*(\varepsilon) = Qt/(Q - \varepsilon t)$ .

If  $u$  is locally integrable,  $(g_j) \in \mathbb{D}^s(u)$  and  $0 < \varepsilon < \varepsilon' < s < \infty$ , then (3.3) with  $t = Q/(Q + \varepsilon)$  and the Hölder's inequality imply that for  $p \geq Q/(Q + \varepsilon)$ ,

$$(3.4) \quad \int_{B(x, 2^{-k})} |u - u_{B(x, 2^{-k})}| d\mu \leq C 2^{-k\varepsilon'} \sum_{j \geq k-2} 2^{-j(s-\varepsilon')} \left( \int_{B(x, 2^{-k+1})} g_j^p d\mu \right)^{1/p}.$$

While working with the Hajlasz–Triebel–Lizorkin spaces  $M_{p,q}^s(X)$  we often use an embedding of these spaces into the space  $M^{s,p}(X)$  and employ the following Sobolev–Poincaré inequality for  $s$ -gradients.

**Lemma 3.4** ([11], Lemma 2.2). *Let  $0 < s < \infty$  and  $0 < t < Q/s$ . There exists a constant  $C > 0$  such that for all measurable functions  $u$  with  $g \in \mathcal{D}^s(u)$ ,  $x \in X$  and  $r > 0$ ,*

$$(3.5) \quad \inf_{c \in \mathbb{R}} \left( \int_{B(x, r)} |u(y) - c|^{t^*(s)} d\mu(y) \right)^{1/t^*(s)} \leq C r^s \left( \int_{B(x, 2r)} g^t d\mu \right)^{1/t},$$

where  $t^*(s) = Qt/(Q - st)$ .

For  $s = 1$  inequality (3.5) is given by [13, Thm 8.7], as well as for  $s \in (0, 1)$ , since in this case  $d^s$  is a distance in  $X$ .

**3.2. Median values.** Using integral averages of a function is a standard technique in construction an extension operator for a locally integrable function. Since we are dealing with the  $L^p$ -integrable functions, possibly with  $0 < p < 1$ , it is convenient to replace in the argument the integral averages by the median values, as for example in [48]. This allows to handle in the same way spaces of functions with the integrability parameter  $0 < p < \infty$ ; a certain disadvantage of this uniform treatment is that the resulting extension operator appears to be non-linear.

**Definition 3.5.** The median value of a measurable function  $u$  on a set  $A \subset X$  is

$$(3.6) \quad m_u(A) = \max_{a \in \mathbb{R}} \left\{ \mu(\{x \in A : u(x) < a\}) \leq \frac{\mu(A)}{2} \right\}.$$

The following properties of medians justify their role of counterparts for the integral averages in the context.

**Lemma 3.6** ([48], Lemma 2.2; [11], (2.4)). *Let  $0 < \eta \leq 1$  and  $u \in L_{loc}^\eta(X)$ . Then*

$$(3.7) \quad |m_u(B) - c| \leq \left( 2 \int_B |u - c|^\eta d\mu \right)^{1/\eta}.$$

for all balls  $B$  and all  $c \in \mathbb{R}$ . Moreover,

$$(3.8) \quad u(x) = \lim_{r \rightarrow 0} m_u(B(x, r))$$

at every Lebesgue point  $x \in X$ .

**Remark 3.7.** Property (3.8) follows from (3.7) by the Lebesgue differentiation theorem. The proof of [48, Lemma 2.2] shows that inequality (3.7) holds for all measurable sets  $E$  with positive and finite measure. In particular, (3.7) holds for every set  $B \cap S$ , where  $S$  satisfies measure density condition (1.1) and  $B$  is a ball centered at  $S$ . This, together with the measure density condition and the Lebesgue differentiation theorem, implies that,

$$u(x) = \lim_{r \rightarrow 0} m_u(B(x, r) \cap S),$$

for almost all  $x \in S$ .

By combining (3.7) and Lemma 3.3, we obtain the following result, which is frequently used in the proof of Theorem 1.2.

**Lemma 3.8.** *Let  $0 < t < \infty$  and  $0 < \varepsilon' < s < 1$ . Let  $k \in \mathbb{Z}$ ,  $x \in X$  and let  $B$  be a ball such that  $B \subset B(x, 2^{-k})$  and  $\mu(B) \approx \mu(B(x, 2^{-k}))$ . Then there exists a constant  $C > 0$  such that for all measurable functions  $u$  with  $(g_j) \in \mathbb{D}^s(u)$ ,*

$$(3.9) \quad |m_u(B) - m_u(B(x, 2^{-k}))| \leq C 2^{-k\varepsilon'} \sum_{j \geq k-2} 2^{-j(s-\varepsilon')} \left( \int_{B(x, 2^{-k+1})} g_j^t d\mu \right)^{1/t}.$$

*Proof.* Let  $0 < \varepsilon < \varepsilon'$  and let  $\eta = t^*(\varepsilon) = Qt/(Q - \varepsilon t)$  if  $t \leq Q/(Q + \varepsilon)$ , and  $\eta = 1$  otherwise. Using (3.3) and the Hölder inequality, we obtain

(3.10)

$$\inf_{c \in \mathbb{R}} \left( \int_{B(x, 2^{-k})} |u(y) - c|^\eta d\mu(y) \right)^{1/\eta} \leq C 2^{-k\varepsilon'} \sum_{j \geq k-2} 2^{-j(s-\varepsilon')} \left( \int_{B(x, 2^{-k+1})} g_j^t d\mu \right)^{1/t}.$$

Let  $c \in \mathbb{R}$ . By (3.7), we have

$$\begin{aligned} |m_u(B) - m_u(B(x, 2^{-k}))| &\leq |m_u(B) - c| + |c - m_u(B(x, 2^{-k}))| \\ &\leq \left( 2 \int_B |u - c|^\eta d\mu \right)^{1/\eta} + \left( 2 \int_{B(x, 2^{-k})} |u - c|^\eta d\mu \right)^{1/\eta} \\ &\leq C \left( \int_{B(x, 2^{-k})} |u - c|^\eta d\mu \right)^{1/\eta}. \end{aligned}$$

The claim follows by taking the infimum over  $c \in \mathbb{R}$  and applying (3.10).  $\square$

**Remark 3.9.** If a set  $S$  satisfies measure density condition (1.1), then the induced space  $(S, d, \mu|_S)$  satisfies doubling condition (2.1) locally, that is, for small radii, and we can replace small balls  $B$ , which are centered in  $S$ , with  $B \cap S$  in inequality (3.9).

**3.3. Leibniz type rules and norm estimates for Lipschitz functions.** We finish this section by proving a Leibniz type rule for fractional  $s$ -gradients and some norm estimates for Lipschitz functions. These norm estimates are used later to show that the extension property for Besov spaces, or for Triebel–Lizorkin spaces, implies measure density condition (1.1).

**Lemma 3.10.** *Let  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ , and let  $S \subset X$  be a measurable set. Let  $u: X \rightarrow \mathbb{R}$  be a measurable function with  $(g_k) \in \mathbb{D}^s(u)$  and let  $\varphi$  be a bounded  $L$ -Lipschitz function supported in  $S$ . Then sequences  $(h_k)_{k \in \mathbb{Z}}$  and  $(\rho_k)_{k \in \mathbb{Z}}$ , where*

$$\rho_k = (g_k \|\varphi\|_\infty + 2^{k(s-1)} L |u|) \chi_{\text{supp } \varphi} \quad \text{and} \quad h_k = (g_k + 2^{sk+2} |u|) \|\varphi\|_\infty \chi_{\text{supp } \varphi}$$

*are fractional  $s$ -gradients of  $u\varphi$ . Moreover, if  $u \in M_{p,q}^s(S)$ , then  $u\varphi \in M_{p,q}^s(X)$  and  $\|u\varphi\|_{M_{p,q}^s(X)} \leq C \|u\|_{M_{p,q}^s(S)}$ .*

*Proof.* For the first claim, let  $x, y \in X$ , and let  $k \in \mathbb{Z}$  such that  $2^{-k-1} \leq d(x, y) < 2^{-k}$ . By the triangle inequality, we have

$$|u(x)\varphi(x) - u(y)\varphi(y)| \leq |u(x)| |\varphi(x) - \varphi(y)| + |\varphi(y)| |u(x) - u(y)|.$$

We consider four cases depending whether  $x$  or  $y$  belongs to  $\text{supp } \varphi$  or not. If  $x, y \in \text{supp } \varphi$ , then

$$\begin{aligned} |u(x)\varphi(x) - u(y)\varphi(y)| &\leq |u(x)| L d(x, y) + \|\varphi\|_\infty d(x, y)^s (g_k(x) + g_k(y)) \\ &\leq d(x, y)^s (2^{k(s-1)} L |u(x)| + \|\varphi\|_\infty (g_k(x) + g_k(y))) \\ &\leq d(x, y)^s (\rho_k(x) + \rho_k(y)), \end{aligned}$$

and, on the other hand,

$$\begin{aligned} |u(x)\varphi(x) - u(y)\varphi(y)| &\leq 2\|\varphi\|_\infty|u(x)| + \|\varphi\|_{L^\infty}d(x,y)^s(g_k(x) + g_k(y)) \\ &\leq d(x,y)^s\|\varphi\|_\infty(2 \cdot 2^{s(k+1)}|u(x)| + g_k(x) + g_k(y)) \\ &\leq d(x,y)^s(h_k(x) + h_k(y)). \end{aligned}$$

Hence, in this case,  $(\rho_k)_{k \in \mathbb{Z}}$  and  $(h_k)_{k \in \mathbb{Z}}$  satisfy the required inequality. The remaining two cases are considered in the same, even simpler, way. This shows that  $(\rho_k)_{k \in \mathbb{Z}}$  and  $(h_k)_{k \in \mathbb{Z}}$  are fractional  $s$ -gradients of  $u\varphi$ .

To prove the second claim, suppose that  $\|\vec{g}\|_{L^p(S, l^q)} \leq 2 \inf \|\vec{r}\|_{L^p(S, l^q)}$ , where the infimum is taken over fractional  $s$ -gradients of  $u$  in  $S$ . By the first part of the proof, the sequence  $(g'_k)_{k \in \mathbb{Z}}$ ,

$$g'_k = \begin{cases} h_k, & \text{if } k < k_L, \\ \rho_k, & \text{if } k \geq k_L, \end{cases}$$

where  $k_L$  is an integer such that  $2^{k_L-1} < L \leq 2^{k_L}$ , is a fractional  $s$ -gradient of  $u\varphi$ .

Concerning the norm, if  $0 < q < \infty$ , we have

$$\begin{aligned} \left( \sum_{k \in \mathbb{Z}} |g'_k|^q \right)^{1/q} &\leq C \left( \|\varphi\|_\infty \left( \sum_{k=-\infty}^{k_L-1} (g_k + 2^{sk+2}|u|)^q \right)^{1/q} \right. \\ &\quad \left. + \left( \sum_{k=k_L}^{\infty} (g_k \|\varphi\|_\infty + 2^{k(s-1)}L|u|)^q \right)^{1/q} \right) \\ &\leq C \left( \|\varphi\|_\infty \left( \sum_{k \in \mathbb{Z}} |g_k|^q \right)^{1/q} + |u| \left( \sum_{k=-\infty}^{k_L-1} 2^{(sk+2)q} \right)^{1/q} \right. \\ &\quad \left. + L|u| \left( \sum_{k=k_L}^{\infty} 2^{kq(s-1)} \right)^{1/q} \right), \end{aligned}$$

and hence

$$\begin{aligned} \|\vec{g}'\|_{L^p(X, l^q)} &\leq C(\|\varphi\|_\infty \|\vec{g}\|_{L^p(S, l^q)} + \|u\|_{L^p(S)} 2^{sk_L} + L\|u\|_{L^p(S)} 2^{k_L(s-1)}) \\ &\leq C(\|\varphi\|_\infty \|\vec{g}\|_{L^p(S, l^q)} + L^s \|u\|_{L^p(S)}). \end{aligned}$$

The claim follows by the selection of  $(g_k)_{k \in \mathbb{Z}}$ . The case  $q = \infty$  follows using similar arguments.  $\square$

**Remark 3.11.** An analogue of Lemma 3.10 holds also for functions from Hajlasz–Besov spaces  $N_{p,q}^s(S)$ . To prove this, it remains to show that  $\|\vec{g}'\|_{l^q(X, L^p)} < \infty$ , with the corresponding bound for the norm, whenever  $\vec{g} \in \mathbb{D}^s(u)$  is such that  $\|\vec{g}\|_{l^q(S, L^p)} < \infty$ .

Indeed, when  $0 < q < \infty$ , we have

$$\begin{aligned}
\|\vec{g}'\|_{l^q(X, L^p)} &= \left( \sum_{k \in \mathbb{Z}} \|g'_k\|_{L^p(X)}^q \right)^{1/q} \\
&= \|\varphi\|_{L^\infty} \left( \sum_{k=-\infty}^{k_L-1} \|g_k + 2^{sk+2}|u|\|_{L^p(S)}^q \right)^{1/q} \\
&\quad + \left( \sum_{k=k_L}^{\infty} \|g_k\|_{L^\infty} + 2^{k(s-1)}L\|u\|_{L^p(S)}^q \right)^{1/q} \\
&\leq C\|\varphi\|_{L^\infty} \left( \sum_{k \in \mathbb{Z}} \|g_k\|_{L^p(S)}^q \right)^{1/q} + \|u\|_{L^p(S)} \left( \sum_{k=-\infty}^{k_L-1} 2^{(sk+2)q} \right)^{1/q} \\
&\quad + L\|u\|_{L^p(S)} \left( \sum_{k=k_L}^{\infty} 2^{kq(s-1)} \right)^{1/q} \\
&\leq C\|\varphi\|_{L^\infty} \|\vec{g}'\|_{l^q(S, L^p)} + L^s\|u\|_{L^p(S)},
\end{aligned}$$

which implies the claim. The case  $q = \infty$  follows similarly.

By selecting  $u \equiv 1$  and  $g_k \equiv 0$  for all  $k \in \mathbb{Z}$  in (the proof of) Lemma 3.11, we obtain norm estimates for Lipschitz functions supported in bounded sets.

**Corollary 3.12.** *Let  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ . Let  $\Omega \subset X$  be a measurable set and let  $\varphi: \Omega \rightarrow \mathbb{R}$  be an  $L$ -Lipschitz function supported in a bounded set  $F \subset \Omega$ . Then  $\varphi \in M_{p,q}^s(\Omega)$  and*

$$(3.11) \quad \|\varphi\|_{M_{p,q}^s(\Omega)} \leq C(1 + \|\varphi\|_{L^\infty})(1 + L^s)\mu(F)^{1/p},$$

where the constant  $C > 0$  depends only on  $s$  and  $q$ . The claim holds also with  $M_{p,q}^s(\Omega)$  replaced by  $N_{p,q}^s(\Omega)$ .

#### 4. INTERPOLATION AND EMBEDDING THEOREMS FOR BESOV SPACES

In this section, we prove new interpolation and embedding theorems for Besov spaces. Recall some essential definitions and properties of the real interpolation theory; see, for example, the classical references [1], [2] for the details.

Let  $A_0$  and  $A_1$  be (quasi-)normed spaces continuously embedded into a topological vector space  $\mathcal{A}$ . For every  $f \in A_0 + A_1$  and  $t > 0$ , the  $K$ -functional is

$$K(f, t; A_0, A_1) = \inf \{ \|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1 \}.$$

Let  $0 < s < 1$  and  $0 < q \leq \infty$ . The interpolation space  $(A_0, A_1)_{s,q}$  consists of functions  $f \in A_0 + A_1$ , for which

$$\|f\|_{(A_0, A_1)_{s,q}} = \begin{cases} \left( \int_0^\infty (t^{-s} K(f, t; A_0, A_1))^q \frac{dt}{t} \right)^{1/q}, & \text{if } q < \infty \\ \sup_{t>0} t^{-s} K(f, t; A_0, A_1), & \text{if } q = \infty, \end{cases}$$

is finite.



The following theorem is the main result of this section. We will apply it later only in the case  $q = \infty$ , but since this interpolation result is of independent interest, we prove it in full generality. The case  $1 \leq p < \infty$ ,  $1 \leq q \leq \infty$  was earlier obtained in [9, Cor. 4.3] using a version of the Korevaar–Schoen definition for the Sobolev spaces in the metric setting.

**Theorem 4.1.** *Let  $X$  be a metric space with a doubling measure  $\mu$ . Let  $0 < p < \infty$ ,  $0 < q \leq \infty$  and  $0 < s < 1$ . Then*

$$(4.1) \quad \dot{N}_{p,q}^s(X) = (L^p(X), \dot{M}^{1,p}(X))_{s,q}$$

and

$$(4.2) \quad N_{p,q}^s(X) = (L^p(X), M^{1,p}(X))_{s,q}$$

with equivalent norms.

*Proof.* To prove (4.1), we first show that there exists a constant  $C > 0$  such that for all  $t > 0$ ,

$$(4.3) \quad C^{-1}E_p(f, t) \leq K(f, t; L^p(X), \dot{M}^{1,p}(X)) \leq C \left( \sum_{k=0}^{\infty} 2^{-k\tilde{p}} E_p^{\tilde{p}}(f, 2^k t) \right)^{1/\tilde{p}},$$

where  $\tilde{p} = \min\{p, 1\}$  and  $E_p(f, t)$  is as in (2.6).

We begin with the first inequality in (4.3). Let  $f = g + h$ , where  $g \in L^p(X)$  and  $h \in \dot{M}^{1,p}(X)$ , and let  $t > 0$ . Then

$$E_p(f, t) \leq C(E_p(g, t) + E_p(h, t)),$$

where, by the Fubini theorem,

$$(4.4) \quad \begin{aligned} E_p^p(g, t) &\leq 2^p \int_X |g(x)|^p d\mu(x) + 2^p \int_X \int_{B(x,t)} |g(y)|^p d\mu(y) d\mu(x) \\ &= 2^p \int_X |g(x)|^p d\mu(x) + 2^p \int_X |g(y)|^p \int_{B(y,t)} \frac{1}{\mu(B(x,t))} d\mu(x) d\mu(y) \\ &\leq C \|g\|_{L^p(X)}^p. \end{aligned}$$

The last estimate follows using the doubling property of  $\mu$  and the fact that  $B(y, t) \subset B(x, 2t)$  for each  $x \in B(y, t)$ .

By the definition of the 1-gradient and by the similar argument as in (4.4), for every  $\rho \in \mathcal{D}(h) \cap L^p(X)$ , we have,

$$\begin{aligned} E_p^p(h, t) &= \int_X \int_{B(x,t)} |h(x) - h(y)|^p d\mu(y) d\mu(x) \\ &\leq \int_X \int_{B(x,t)} (d(x, y))^p (\rho(x) + \rho(y))^p d\mu(y) d\mu(x) \\ &\leq Ct^p \left( \int_X \rho(x)^p d\mu(x) + \int_X \int_{B(x,t)} \rho(y)^p d\mu(y) d\mu(x) \right) \\ &\leq Ct^p \|\rho\|_{L^p(X)}^p. \end{aligned}$$

By taking the infimum over all representations of  $f$  in  $L^p(X) + \dot{M}^{1,p}(X)$ , we have that  $E_p(f, t) \leq CK(f, t; L^p(X), \dot{M}^{1,p}(X))$ .

To prove the second inequality in (4.3), let  $f \in L^p_{\text{loc}}(X)$  and let  $t > 0$ . By a standard covering argument, there is a covering of  $X$  by balls  $B_i = B(x_i, t/6)$ ,  $i \in \mathbb{N}$ , such that  $\sum_i \chi_{2B_i} \leq N$  with the overlap constant  $N > 0$  depending only on the doubling constant of  $\mu$ .

Let  $\{\varphi_i\}_{i \in \mathbb{N}}$  be a collection of  $Ct^{-1}$ -Lipschitz functions  $\varphi_i: X \rightarrow [0, 1]$  such that  $\text{supp } \varphi_i \subset 2B_i$  and  $\sum_i \varphi_i(x) = 1$  for all  $x \in X$ , (this is a so called partition of unity subordinate to the covering  $\{B_i\}_{i \in \mathbb{N}}$ , see also the beginning of Section 5).

Let  $h: X \rightarrow \mathbb{R}$  be a function defined using median values (3.6) of  $f$ ,

$$h(x) = \sum_{i \in \mathbb{N}} m_f(B_i) \varphi_i(x), \quad \text{for all } x \in X,$$

and let  $g = f - h$ .

Let  $x \in X$  and let  $I_x = \{i : x \in 2B_i\}$ . By the properties of the partition of unity,

$$g(x) = \sum_{i \in \mathbb{N}} (f(x) - m_f(B_i)) \varphi_i(x) = \sum_{i \in I_x} (f(x) - m_f(B_i)) \varphi_i(x).$$

Since the number of elements in  $I_x$  is bounded by the overlap constant  $N$  independent of  $x$  and  $t$ , and, for every  $i \in I_x$ ,  $B_i \subset B(x, t) \subset 8B_i$ , using (3.7) and the doubling property of  $\mu$ , we obtain

$$(4.5) \quad |g(x)| \leq 2 \sum_{i \in I_x} \left( \int_{B_i} |f(x) - f(z)|^p d\mu(z) \right)^{1/p} \leq C \left( \int_{B(x, t)} |f(x) - f(z)|^p d\mu(z) \right)^{1/p},$$

which implies that

$$(4.6) \quad \|g\|_{L^p(X)} \leq CE_p(f, t).$$

Next we estimate  $h$  in the  $\dot{M}^{1,p}$ -norm. Let  $t > 0$  and let  $x, y \in X$ . We consider two cases.

CASE 1: If  $d(x, y) \leq t$ , then  $B_i \subset B(x, 2t) \subset 20B_i$ , for every  $i \in I_x \cup I_y$ . Using the properties of the functions  $\varphi_i$ , we have

$$\begin{aligned} h(x) - h(y) &= \sum_{i \in \mathbb{N}} (m_f(B_i) - f(x))(\varphi_i(x) - \varphi_i(y)) \\ &= \sum_{i \in I_x \cup I_y} (m_f(B_i) - f(x))(\varphi_i(x) - \varphi_i(y)), \end{aligned}$$

which together with the  $Ct^{-1}$ -Lipschitz continuity of the functions  $\varphi_i$  and (3.7) implies that

$$\begin{aligned} |h(x) - h(y)| &\leq C \frac{d(x, y)}{t} \sum_{i \in I_x \cup I_y} \left( \int_{B_i} |f(x) - f(z)|^p d\mu(z) \right)^{1/p} \\ &\leq C \frac{d(x, y)}{t} \left( \int_{B(x, 2t)} |f(x) - f(z)|^p d\mu(z) \right)^{1/p}. \end{aligned}$$

CASE 2: Let  $d(x, y) > t$ . Since

$$|h(x) - h(y)| \leq |f(x) - f(y)| + |g(x)| + |g(y)|,$$

it suffices to estimate the terms on the right side. The assumption  $d(x, y) > t$  and (4.5) imply that

$$|g(x)| \leq C \frac{d(x, y)}{t} \left( \int_{B(x, t)} |f(z) - f(x)|^p d\mu(z) \right)^{1/p},$$

and a corresponding upper bound holds for  $|g(y)|$ .

Using (3.7) and the doubling property of  $\mu$  and writing  $R = d(x, y)$ , we obtain

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - m_f(B(x, R))| + |f(y) - m_f(B(x, R))| \\ &\leq 2 \left( \int_{B(x, R)} |f(z) - f(x)|^p d\mu(z) \right)^{1/p} \\ &\quad + C \left( \int_{B(y, 2R)} |f(z) - f(y)|^p d\mu(z) \right)^{1/p}, \end{aligned}$$

and, hence,

$$|f(x) - f(y)| \leq Cd(x, y)(f_t^\sharp(x) + f_t^\sharp(y)),$$

where

$$f_t^\sharp(x) = \sup_{r \geq t} \frac{1}{r} \left( \int_{B(x, r)} |f(z) - f(x)|^p d\mu(z) \right)^{1/p}.$$

Collecting the estimates, we obtain, in both cases, that

$$|h(x) - h(y)| \leq Cd(x, y)(f_t^\sharp(x) + f_t^\sharp(y)),$$

which shows that  $f_t^\sharp \in \mathcal{D}(h)$ . Hence it suffices to estimate  $\|f_t^\sharp\|_{L^p(X)}$ .

Using the definition of  $f_t^\sharp$  and the doubling property of  $\mu$ , we have

$$\begin{aligned} f_t^\sharp(x) &\leq \sum_{k=0}^{\infty} \sup_{2^{k-1}t < r \leq 2^k t} \frac{1}{r} \left( \int_{B(x, r)} |f(z) - f(x)|^p d\mu(z) \right)^{1/p} \\ &\leq \frac{C}{t} \sum_{k=0}^{\infty} 2^{-k} \left( \int_{B(x, 2^k t)} |f(z) - f(x)|^p d\mu(z) \right)^{1/p}. \end{aligned}$$

If  $0 < p \leq 1$ , we use inequality (3.1) and obtain

$$\begin{aligned} \|f_t^\sharp\|_{L^p(X)}^p &\leq \frac{C}{t^p} \sum_{k=0}^{\infty} 2^{-kp} \int_X \int_{B(x, 2^k t)} |f(z) - f(x)|^p d\mu(z) d\mu(x) \\ (4.7) \quad &= \frac{C}{t^p} \sum_{k=0}^{\infty} 2^{-kp} E_p^p(f, 2^k t). \end{aligned}$$

When  $p > 1$ , we have, by the Minkowski inequality, that

$$(4.8) \quad \|f_t^\sharp\|_{L^p(X)} \leq \frac{C}{t} \sum_{k=0}^{\infty} 2^{-k} E_p(f, 2^k t).$$

Thus, the required inequality

$$K(f, t; L^p(X), \dot{M}^{1,p}(X)) \leq C \left( \sum_{k=0}^{\infty} 2^{-k\tilde{p}} E_p^{\tilde{p}}(f, 2^k t) \right)^{1/\tilde{p}}$$

follows using (4.6)-(4.8) and the definition of the  $K$ -functional.

**Interpolation result (4.1) for  $\dot{N}_{p,q}^s(X)$ :** The equivalence of Besov norms (2.7), the definition of the norm  $\|f\|_{\dot{B}_{p,q}^s(X)}$  and the first inequality in (4.3) imply that

$$\|f\|_{\dot{N}_{p,q}^s(X)} \leq C \|f\|_{(L^p(X), \dot{M}^{1,p}(X))_{s,q}}.$$

To obtain the opposite estimate, we use the second inequality of (4.3).

If  $q < \infty$ , then

$$\begin{aligned} \|f\|_{(L^p(X), \dot{M}^{1,p}(X))_{s,q}} &= \left( \int_0^\infty (t^{-s} K(f, t; L^p(X), \dot{M}^{1,p}(X)))^q \frac{dt}{t} \right)^{1/q} \\ &\leq C \left( \int_0^\infty \left( \sum_{k=0}^{\infty} t^{-s\tilde{p}} 2^{-k\tilde{p}} E_p^{\tilde{p}}(f, 2^k t) \right)^{q/\tilde{p}} \frac{dt}{t} \right)^{1/q}, \end{aligned}$$

where the last estimate is denoted by  $A$ . If  $q \leq \tilde{p}$ , then using (3.1) and change of variables, we obtain

$$\begin{aligned} A &\leq \left( \sum_{k=0}^{\infty} 2^{-kq} \int_0^\infty (t^{-s} E_p(f, 2^k t))^q \frac{dt}{t} \right)^{1/q} \\ &= \left( \sum_{k=0}^{\infty} 2^{-kq} \int_0^\infty ((2^{-k}\tau)^{-s} E_p(f, \tau))^q \frac{d\tau}{\tau} \right)^{1/q} \\ &= \left( \sum_{k=0}^{\infty} 2^{kq(s-1)} \right)^{1/q} \left( \int_0^\infty (\tau^{-s} E_p(f, \tau))^q \frac{d\tau}{\tau} \right)^{1/q}. \end{aligned}$$

If  $q \geq \tilde{p}$ , then, using the Minkowski inequality and changing variables, we have

$$\begin{aligned} A &\leq \left( \sum_{k=0}^{\infty} \left( \int_0^\infty (t^{-s\tilde{p}} 2^{-k\tilde{p}} E_p^{\tilde{p}}(f, 2^k t))^{q/\tilde{p}} \frac{dt}{t} \right)^{\tilde{p}/q} \right)^{1/\tilde{p}} \\ &= \left( \sum_{k=0}^{\infty} \left( \int_0^\infty ((2^{-k}\tau)^{-s\tilde{p}} 2^{-k\tilde{p}} E_p^{\tilde{p}}(f, \tau))^{q/\tilde{p}} \frac{d\tau}{\tau} \right)^{\tilde{p}/q} \right)^{1/\tilde{p}} \\ &= \left( \sum_{k=0}^{\infty} 2^{k\tilde{p}(s-1)} \right)^{1/\tilde{p}} \left( \int_0^\infty (\tau^{-s} E_p(f, \tau))^q \frac{d\tau}{\tau} \right)^{1/q}. \end{aligned}$$

In the case  $q = \infty$ , we have, using (4.3),

$$\begin{aligned}
\|f\|_{(L^p(X), \dot{M}^{1,p}(X))_{s,\infty}} &= \sup_{t>0} t^{-s} K(f, t; L^p(X), \dot{M}^{1,p}(X)) \\
&\leq C \sup_{t>0} t^{-s} \left( \sum_{k=0}^{\infty} 2^{-k\tilde{p}} E_p^{\tilde{p}}(f, 2^k t) \right)^{1/\tilde{p}} \\
&= C \sup_{t>0} \left( \sum_{k=0}^{\infty} 2^{k\tilde{p}(s-1)} \left( (2^k t)^{-s} E_p(f, 2^k t) \right)^{\tilde{p}} \right)^{1/\tilde{p}} \\
&\leq C \left( \sum_{k=0}^{\infty} 2^{k\tilde{p}(s-1)} \right)^{1/\tilde{p}} \sup_{t>0} t^{-s} E_p(f, t).
\end{aligned}$$

By the comparability of the norms in Besov spaces given by different definitions, (2.7), we have

$$\|f\|_{(L^p(X), \dot{M}^{1,p}(X))_{s,q}} \leq C \|f\|_{\dot{N}_{p,q}^s(X)}.$$

**Interpolation result (4.2) for  $N_{p,q}^s(X)$ :** We start by showing that

$$(4.9) \quad K(f, t, L^p(X), M^{1,p}(X)) \approx K(f, t; L^p(X), \dot{M}^{1,p}(X)) + \min\{1, t\} \|f\|_{L^p(X)}$$

for each  $f \in L^p(X) + M^{1,p}(X)$  and every  $t > 0$ .

Let  $f$  be such a function and let  $t > 0$ . The definition of the  $K$ -functional and the spaces  $\dot{M}^{1,p}(X)$  and  $M^{1,p}(X)$  imply that

$$K(f, t; L^p(X), \dot{M}^{1,p}(X)) \leq K(f, t, L^p(X), M^{1,p}(X)).$$

For every  $g \in L^p(X)$  and  $h \in M^{1,p}(X)$  with  $f = g + h$ , we have

$$\begin{aligned}
\min\{1, t\} \|f\|_{L^p(X)} &\leq C \left( \min\{1, t\} \|g\|_{L^p(X)} + \min\{1, t\} \|h\|_{L^p(X)} \right) \\
&\leq C \left( \|g\|_{L^p(X)} + t \|h\|_{M^{1,p}(X)} \right),
\end{aligned}$$

which implies that

$$\min\{1, t\} \|f\|_{L^p(X)} \leq C K(f, t, L^p(X), M^{1,p}(X)).$$

This implies one direction of inequality (4.9).

For the other direction, assume first that  $t > 1$ . Then the claim follows from the fact that

$$K(f, t, L^p(X), M^{1,p}(X)) \leq \|f\|_{L^p(X)}.$$

If  $0 < t < 1$ , let  $g \in L^p(X)$  and  $h \in \dot{M}^{1,p}(X)$  be such that  $f = g + h$ . Then  $h \in L^p(X)$  with  $\|h\|_{L^p(X)} \leq C(\|f\|_{L^p(X)} + \|g\|_{L^p(X)})$  and

$$\begin{aligned}
K(f, t, L^p(X), M^{1,p}(X)) &\leq \|g\|_{L^p(X)} + t \|h\|_{M^{1,p}(X)} \\
&= \|g\|_{L^p(X)} + t \|h\|_{L^p(X)} + t \|h\|_{\dot{M}^{1,p}(X)} \\
&\leq C \left( \|g\|_{L^p(X)} + t \|h\|_{\dot{M}^{1,p}(X)} + t \|f\|_{L^p(X)} \right),
\end{aligned}$$

from which the claim follows by taking the infimum over such decompositions  $f = g + h$ .

Since

$$\int_0^\infty (t^{-s} \min\{1, t\})^q \frac{dt}{t} < \infty \quad \text{and} \quad \sup_{t>0} t^{-s} \min\{1, t\} < \infty,$$

(4.9) implies that

$$\|f\|_{(L^p(X), M^{1,p}(X))_{s,q}} \approx \|f\|_{(L^p(X), \dot{M}^{1,p}(X))_{s,q}} + \|f\|_{L^p(X)},$$

and, hence, (4.2) follows from (4.1).  $\square$

**Remark 4.2.** Since each linear operator which is bounded in  $L^p$  and in  $M^{1,p}$ , is bounded in the interpolation space, the extension results for Besov space with  $p \geq 1$  follow from Theorem 4.1 and the extension results in [15].

Theorem 4.1 and the reiteration theorem [20, Thm 3.1] imply the following interpolation theorem for Hajłasz–Besov spaces. In the Euclidean setting, this result was proved in [6] using different methods. For related interpolation results in the metric setting, see [46], [16] and [9].

**Theorem 4.3.** *Let  $X$  be a metric space with a doubling measure  $\mu$ . Let  $0 < p < \infty$ ,  $0 < q, q_0, q_1 \leq \infty$ ,  $0 < s_0, s_1, \lambda < 1$ , and  $s = (1 - \lambda)s_0 + \lambda s_1$ . Then*

$$(\dot{N}_{p,q_0}^{s_0}(X), \dot{N}_{p,q_1}^{s_1}(X))_{\lambda,q} = \dot{N}_{p,q}^s(X)$$

and

$$(N_{p,q_0}^{s_0}(X), N_{p,q_1}^{s_1}(X))_{\lambda,q} = N_{p,q}^s(X)$$

with equivalent norms.

**4.1. An Embedding theorem for the Hajłasz–Besov spaces.** Our interpolation theorem implies a Sobolev type embedding result for the Hajłasz–Besov spaces. The embedding is into the Lorentz spaces.

Recall that the Lorentz space  $L^{p,q}(X)$ ,  $0 < p < \infty$ ,  $0 < q \leq \infty$ , consists of measurable functions  $u: X \rightarrow [-\infty, \infty]$ , for which the (quasi)norm

$$\|u\|_{L^{p,q}(X)} = p^{1/q} \left( \int_0^\infty t^q \mu(\{x \in X : |u(x)| \geq t\})^{q/p} \frac{dt}{t} \right)^{1/q},$$

when  $q < \infty$ , and

$$\|u\|_{L^{p,\infty}(X)} = \sup_{t>0} t \mu(\{x \in X : |u(x)| > t\})^{1/p},$$

when  $q = \infty$ , is finite. Using the Cavalieri principle, it is easy to see that  $L^{p,p}(X) = L^p(X)$ . Moreover,  $L^{p,\infty}(X)$  equals weak  $L^p(X)$ -space and  $L^{p,q}(X) \subset L^{p,r}(X)$  when  $r > q$ .

In the Euclidean setting, embedding  $\mathcal{B}_{p,q}^s(\mathbb{R}^n) \hookrightarrow L^{p^*(s),q}(\mathbb{R}^n)$  was obtained in [17, Thm 1.15] using an atomic decomposition of  $\mathcal{B}_{p,q}^s(\mathbb{R}^n)$ . In the metric case, the embedding of Besov spaces  $\mathcal{B}_{p,q}^s(X)$ ,  $p > 1$ ,  $q \geq 1$ , to Lorentz spaces was proved in [9] under the assumption that  $X$  supports a  $(1, p)$ -Poincaré inequality. The idea of our proof comes from [9, Thm 5.1]. For the readers' convenience, we give the proof with all details.



**Theorem 4.4.** *Let  $X$  be a  $Q$ -regular metric space,  $Q \geq 1$ . Let  $0 < s < 1$ ,  $0 < p < Q/s$  and  $0 < q \leq \infty$ . There is a constant  $C > 0$  such that*

$$(4.10) \quad \inf_{c \in \mathbb{R}} \|u - c\|_{L^{p^*(s),q}(X)} \leq C \|u\|_{\dot{N}_{p,q}^s(X)},$$

where  $p^*(s) = Qp/(Q - sp)$ .

*Proof.* By Lemma 3.4, for every ball  $B(x, r) \subset X$  and for every  $u \in \dot{M}^{1,p}(X)$ , we have

$$\inf_{c \in \mathbb{R}} \left( \int_{B(x,r)} |u(y) - c|^{p^*} d\mu(y) \right)^{1/p^*} \leq Cr \left( \int_{B(x,2r)} g^p d\mu \right)^{1/p},$$

where  $p^* = pQ/(Q - p)$ , whenever  $g \in L^p(X)$  is a 1-gradient of  $u$ . Since the measure  $\mu$  is  $Q$ -regular, we obtain

$$\inf_{c \in \mathbb{R}} \left( \int_{B(x,r)} |u(y) - c|^{p^*} d\mu(y) \right)^{1/p^*} \leq C_0 \left( \int_{B(x,2r)} g^p d\mu \right)^{1/p},$$

where the constant  $C_0 > 0$  is independent of  $r$ . It follows that, for every  $k \geq 1$ , there is  $c_k \in \mathbb{R}$  such that

$$\|u - c_k\|_{L^{p^*}(B(x,k))} \leq 2C_0 \|u\|_{\dot{M}^{1,p}(X)}.$$

Since

$$\|u - c_k\|_{L^{p^*}(B(x,1))} \leq \|u - c_k\|_{L^{p^*}(B(x,k))} \leq 2C_0 \|u\|_{\dot{M}^{1,p}(X)},$$

and  $X$  is  $Q$ -regular, we have that

$$\begin{aligned} |c_k| &\leq c_Q^{1/p^*} (\|u - c_k\|_{L^{p^*}(B(x,1))} + \|u\|_{L^{p^*}(B(x,1))}) \\ &\leq c_Q^{1/p^*} (\|u - c_k\|_{L^{p^*}(B(x,1))} + \|u - c_1\|_{L^{p^*}(B(x,1))} + c_1 c_Q^{1/p^*}) \\ &\leq C \|u\|_{\dot{M}^{1,p}(X)} + C, \end{aligned}$$

where the constant  $C > 0$  does not depend on  $k$ . As a bounded sequence in  $\mathbb{R}$ ,  $(c_k)$  has a subsequence  $(c_{k_j})$  that converges to some  $c \in \mathbb{R}$ .

Now, for a fixed  $m$ , and for each  $k_j \geq m$ , we have

$$\begin{aligned} \|u - c\|_{L^{p^*}(B(x,m))} &\leq C (\|u - c_{k_j}\|_{L^{p^*}(B(x,m))} + \|c_{k_j} - c\|_{L^{p^*}(B(x,m))}) \\ &\leq C (\|u - c_{k_j}\|_{L^{p^*}(B(x,k_j))} + \|c_{k_j} - c\|_{L^{p^*}(B(x,m))}) \\ &\leq C (\|u\|_{\dot{M}^{1,p}(X)} + \mu(B(x,m))^{1/p^*} |c_{k_j} - c|). \end{aligned}$$

By letting first  $j \rightarrow \infty$  and then  $m \rightarrow \infty$ , we conclude that

$$\|u - c\|_{L^{p^*}(X)} \leq C \|u\|_{\dot{M}^{1,p}(X)}.$$

Since  $\frac{1-s}{p} + \frac{s}{p^*} = \frac{1}{p^*(s)}$ , an interpolation theorem from [20, Thm 4.3] together with the fact that  $L^{r,r}(X) = L^r(X)$  for each  $r$ , states that

$$L^{p^*(s),q}(X) = (L^p(X), L^{p^*}(X))_{s,q}.$$

Thus, using Theorem 4.1, we obtain

$$\begin{aligned} \inf_{c \in \mathbb{R}} \|u - c\|_{L^{p^*(s),q}(X)} &\leq C \inf_{c \in \mathbb{R}} \|u - c\|_{(L^p(X), L^{p^*}(X))_{s,q}} \\ &\leq C \|u\|_{(L^p(X), \dot{M}^{1,p}(X))_{s,q}} \approx C \|u\|_{\dot{N}_{p,q}^s(X)}. \end{aligned}$$

□

## 5. THE PROOF OF THEOREM 1.2

In order to prove Theorem 1.2, we use a modification of the Whitney extension method, which has been standard in the study of extension problems starting from work [22]. We start by recalling basic properties of the Whitney covering and the corresponding partition of unity; see, for example, [15]. We also refer to [5, Thm III.1.3] and [28, Lemma 2.9] for the proofs of these properties.

Let  $U \subsetneq X$  be an open set and, for each  $x \in U$ , let

$$r(x) = \text{dist}(x, X \setminus U)/10.$$

There exists a countable family  $\mathcal{B} = \{B_i\}_{i \in I}$  of balls  $B_i = B(x_i, r_i)$ , where  $r_i = r(x_i)$ , such that  $\mathcal{B}$  is a covering of  $U$  and the balls  $1/5B_i$  are disjoint. The next lemma easily follows from the definition of the Whitney covering  $\mathcal{B}$  and the doubling property of the measure  $\mu$ .

**Lemma 5.1.** *Let  $\mathcal{B}$  be a Whitney covering of an open set  $U$ . There is  $M \in \mathbb{N}$  such that for all  $i \in \mathbb{N}$ ,*

- (1)  $5B_i \subset U$ ,
- (2) if  $x \in 5B_i$ , then  $5r_i < \text{dist}(x, X \setminus U) < 15r_i$ ,
- (3) there is  $x_i^* \in X \setminus U$  such that  $d(x_i, x_i^*) < 15r_i$ ,
- (4)  $\sum_{i \in I} \chi_{5B_i}(x) \leq M$  for all  $x \in U$ .

Let  $\{\varphi_i\}_{i \in I}$  be a Lipschitz partition of unity subordinated to the covering  $\mathcal{B}$  with the following properties:

- (i)  $\text{supp } \varphi_i \subset 2B_i$ ,
- (ii)  $\varphi_i(x) \geq M^{-1}$  for all  $x \in B_i$ ,
- (iii) there is a constant  $K > 0$  such that each  $\varphi_i$  is  $Kr_i^{-1}$ -Lipschitz,
- (iv)  $\sum_{i \in I} \varphi_i(x) = \chi_U(x)$ .

Note that if  $5B_i \cap 5B_j \neq \emptyset$ , then  $1/3r_i \leq r_j \leq 3r_i$  and  $d(x_i^*, x_j^*) \leq 80r_i$ , where the points  $x_i^*, x_j^*$  are as in Lemma 5.1 (3).

As it was already mentioned, we construct an extension operator using median values of a function. By this technique, we can prove the result for all  $0 < p < \infty$ , but our extension operator appears to be non-linear. If  $p > Q/(Q + s)$ , a linear extension can be obtained by replacing medians  $m_u(B_i^* \cap S)$  with integral averages  $u_{B_i^* \cap S}$  in the definition of the local extension (5.1). This is easy to show employing (3.4) in the proof; we leave the details to the reader.

**5.1. The proof of Theorem 1.2.** Let  $S \subsetneq X$  be a set satisfying measure density condition (1.1). We may assume that  $S$  is closed, because  $\mu(\overline{S} \setminus S) = 0$  by [40, Lemma 2.1].

Assume first that  $u \in M_{p,q}^s(S)$  and that  $(g_k)_{k \in \mathbb{Z}} \in \mathbb{D}^s(u)$  with

$$\|(g_k)\|_{L^p(S, l^q)} < 2 \inf_{(h_k) \in \mathbb{D}^s(u)} \|(h_k)\|_{L^p(S, l^q)}.$$

Although the functions  $g_k$  are defined on  $S$  only, we identify them with functions defined on  $X$  by assuming that each  $g_k = 0$  on  $X \setminus S$ .

Let  $\mathcal{B} = \{B_i\}_{i \in I}$ ,  $B_i = B(x_i, r_i)$ , be a Whitney covering of  $X \setminus S$  and let  $\{\varphi_i\}_{i \in I}$  be the associated Lipschitz partition of unity. Define  $\mathcal{B}_1 = \{B_i\}_{i \in J}$  as the collection of all balls from  $\mathcal{B}$  with radius less than 1, and note that the measure density condition holds for balls in  $\mathcal{B}_1$ . For each  $i \in J$ , let  $x_i^*$  be "the closest point of  $x_i$  in  $S$ " as in Lemma 5.1, let

$$B_i^* = B(x_i^*, r_i),$$

and for each  $x \in 2B_i$ ,  $i \in J$ , let

$$B_x = B(x, 25r(x)) = B(x, \frac{5}{2} \text{dist}(x, S)).$$

Then  $B_i^* \subset B_x \subset 47B_i^*$  and, by the measure density condition and the doubling property of  $\mu$ ,

$$\mu(B_x) \leq C\mu(B_i^* \cap S).$$

### A local extension to the neighborhood of $S$ :

We will first construct an extension of  $u$  with norm estimates to set

$$V = \{x \in X : \text{dist}(x, S) < 8\}.$$

For each  $x \in V \setminus S$ , let

$$I_x = \{i \in I : x \in 2B_i\}.$$

By Lemma 5.1, the number of elements in  $I_x$  is bounded by  $M$ . Moreover, if  $i \in I \setminus J$ , then  $r_i \geq 1$  and hence  $\text{dist}(2B_i, S) \geq 8r_i \geq 8$ . Thus  $2B_i \cap V = \emptyset$  and  $i \notin I_x$ . Accordingly,  $I_x \subset J$  and therefore

$$\sum_{i \in I_x} \varphi_i(x) = \sum_{i \in I} \varphi_i(x) = \sum_{i \in J} \varphi_i(x) = 1 \quad \text{for } x \in V \setminus S.$$

Define the local extension  $\tilde{E}u$  of  $u$  by

$$(5.1) \quad \tilde{E}u(x) = \begin{cases} u(x), & \text{if } x \in S, \\ \sum_{i \in J} \varphi_i(x) m_u(B_i^* \cap S), & \text{if } x \in X \setminus S. \end{cases}$$

We begin by showing that

$$(5.2) \quad \|\tilde{E}u\|_{L^p(V)} \leq C\|u\|_{L^p(S)}.$$

If  $x \in X \setminus S$ , applying (3.7) with  $0 < \eta < p$ , we obtain

$$\begin{aligned} |\tilde{E}u(x)| &\leq \sum_{i \in I_x} \varphi_i(x) |m_u(B_i^* \cap S)| \leq C \sum_{i \in I_x} \left( \int_{B_i^* \cap S} |u|^\eta d\mu \right)^{1/\eta} \\ &\leq C \left( \int_{B_x} |u|^\eta d\mu \right)^{1/\eta} \leq C (\mathcal{M} u^\eta(x))^{1/\eta}. \end{aligned}$$

Note that in the estimates above, we assumed that  $u$  equals zero outside  $S$ , hence,

$$\int_{B_x} |u| d\mu = \mu(B_x)^{-1} \int_{B_x \cap S} |u| d\mu.$$

Now norm estimate (5.2) follows from the definition of  $\tilde{E}u$  and the boundedness of the Hardy–Littlewood maximal operator in  $L^{p/\eta}$ .

**A fractional  $s$ -gradient for the local extension:**

Let  $0 < \delta < 1 - s$ ,  $0 < \varepsilon' < s$  and  $0 < t < \min\{p, q\}$ . We define the sequence  $(\tilde{g}_k)_{k \in \mathbb{Z}}$ , a candidate for the fractional  $s$ -gradient of  $\tilde{E}u$ , as follows

$$(5.3) \quad \tilde{g}_k(x) = \sum_{j=-\infty}^{k-1} 2^{(j-k)\delta} (\mathcal{M} g_j^t(x))^{1/t} + \sum_{j=k-6}^{\infty} 2^{(k-j)(s-\varepsilon')} (\mathcal{M} g_j^t(x))^{1/t}.$$

We will split the rest of the proof of the theorem into several steps.

**Lemma 5.2.** *There is a constant  $C > 0$  such that  $(C\tilde{g}_k)_{k \in \mathbb{Z}}$ , where functions  $\tilde{g}_k$  are given by formula (5.3), is a fractional  $s$ -gradient of  $\tilde{E}u$ .*

*Proof.* Let  $k \in \mathbb{Z}$  and let  $x, y \in V$  be such that  $2^{-k-1} \leq d(x, y) < 2^{-k}$ . We consider the following four cases:

CASE 1: Since, clearly, almost everywhere on  $S$  the inequality  $g_k \leq \tilde{g}_k$  holds, for almost every  $x, y \in S$ ,

$$|\tilde{E}u(x) - \tilde{E}u(y)| = |u(x) - u(y)| \leq d(x, y)^s (\tilde{g}_k(x) + \tilde{g}_k(y)).$$

CASE 2:  $x \in V \setminus S$ ,  $y \in S$ .

Then  $r(x) = \text{dist}(x, S)/10 < 2^{-k}/10$  and there is  $x^* \in B_x \cap S$  such that  $d(x, x^*) < 15r(x)$ . Let  $m \in \mathbb{Z}$  be such that  $2^{-m-1} \leq 50r(x) < 2^{-m}$  and let

$$B_{x^*} = B(x^*, 2^{-m}).$$

Then  $B_x \subset B_{x^*}$  and  $2B_{x^*} \subset 9B_x$ . Now

$$(5.4) \quad |\tilde{E}u(x) - \tilde{E}u(y)| \leq |\tilde{E}u(x) - m_u(B_{x^*} \cap S)| + |u(y) - m_u(B_{x^*} \cap S)|,$$

and we begin with the first term of (5.4). Using (3.9) and the fact that  $B_i^* \subset B_x \subset B_{x^*}$  with comparable measures, we have

$$\begin{aligned} |\tilde{E}u(x) - m_u(B_{x^*} \cap S)| &= \left| \sum_{i \in I_x} \varphi_i(x) (m_u(B_i^* \cap S) - m_u(B_{x^*} \cap S)) \right| \\ &\leq C 2^{-m\varepsilon'} \sum_{j \geq m-2} 2^{-j(s-\varepsilon')} \left( \int_{B(x^*, 2^{-m+1})} g_j^t d\mu \right)^{1/t}, \end{aligned}$$

where

$$\left( \int_{B(x^*, 2^{-m+1})} g_j^t d\mu \right)^{1/t} \leq C \left( \int_{9B_x} g_j^t d\mu \right)^{1/t} \leq C (\mathcal{M} g_j^t(x))^{1/t}.$$

Since

$$2^{-m} \leq 100r(x) = 10 \text{dist}(x, S) \leq 10d(x, y) < 10 \cdot 2^{-k},$$

we have that  $m \geq k - 4$ , and hence

$$\begin{aligned} |\tilde{E}u(x) - m_u(B_{x^*} \cap S)| &\leq C 2^{-ks} \sum_{j \geq k-6} 2^{(k-j)(s-\varepsilon')} (\mathcal{M} g_j^t(x))^{1/t} \\ &\leq C d(x, y)^s \sum_{j \geq k-6} 2^{(k-j)(s-\varepsilon')} (\mathcal{M} g_j^t(x))^{1/t}. \end{aligned}$$

Next we estimate the second term in (5.4). Let  $l$  be the smallest integer such that  $B(y, 2^{-k}) \subset 2^l B_{x^*}$ . Then  $2^{l+1} B_{x^*} \subset B(x, 17 \cdot 2^{-k})$  and the radius of the ball  $2^{l-1} B_{x^*}$

is at most  $2^{-k+2}$ . Moreover, by the selection of  $l$ , the radius  $2^{l-m}$  of the ball  $2^l B_{x^*}$  is comparable to  $2^{-k}$ ,

$$(5.5) \quad 2^{-k} \leq 2^{-k} + d(x^*, y) \leq 2^{l-m} \leq 2^{-k+3}.$$

We have

$$\begin{aligned} |u(y) - m_u(B_{x^*} \cap S)| &\leq |u(y) - m_u(B(y, 2^{-k}) \cap S)| \\ &\quad + |m_u(B(y, 2^{-k}) \cap S) - m_u(2^l B_{x^*} \cap S)| + |m_u(2^l B_{x^*} \cap S) - m_u(B_{x^*} \cap S)| \\ &= (a) + (b) + (c), \end{aligned}$$

and we estimate the terms  $(a) - (c)$  separately.

Let  $y$  be such that (3.8) holds (almost every point is such a point). Using (3.9) and estimating the geometric series in the third row by its first term  $2^{-k\varepsilon'}$ , we obtain

$$\begin{aligned} (a) &\leq \sum_{i=0}^{\infty} |m_u(B(y, 2^{-i-k}) \cap S) - m_u(B(y, 2^{-(i+1)-k}) \cap S)| \\ &\leq C \sum_{i=0}^{\infty} 2^{(-i-k)\varepsilon'} \sum_{j=(i+k)-2}^{\infty} 2^{-j(s-\varepsilon')} \left( \int_{B(y, 2^{-i-k+1})} g_j^t d\mu \right)^{1/t} \\ &\leq C \sum_{j=k-2}^{\infty} 2^{-j(s-\varepsilon')} (\mathcal{M} g_j^t(y))^{1/t} \sum_{i=0}^{\infty} 2^{(-i-k)\varepsilon'} \\ &\leq C 2^{-k\varepsilon'} \sum_{j=k-2}^{\infty} 2^{-j(s-\varepsilon')} (\mathcal{M} g_j^t(y))^{1/t} \\ &\leq C 2^{-ks} \sum_{j=k-2}^{\infty} 2^{(k-j)(s-\varepsilon')} (\mathcal{M} g_j^t(y))^{1/t} \\ &\leq C d(x, y)^s \sum_{j=k-2}^{\infty} 2^{(k-j)(s-\varepsilon')} (\mathcal{M} g_j^t(y))^{1/t}. \end{aligned}$$

For term (b), we use (3.9) and the fact  $B(y, 2^{-k}) \subset 2^l B_{x^*} \subset 5B(y, 2^{-k})$  to obtain

$$\begin{aligned} (b) &= |m_u(B(y, 2^{-k}) \cap S) - m_u(2^l B_{x^*} \cap S)| \\ &\leq C 2^{-(m-l)\varepsilon'} \sum_{j \geq m-l-2} 2^{-j(s-\varepsilon')} \left( \int_{2^{l+1} B_{x^*}} g_j^t d\mu \right)^{1/t}. \end{aligned}$$

Now (5.5) together with the preceding discussion implies that

$$2^{l+1} B_{x^*} \subset 17B(x, 2^{-k}) \subset C 2^{l+1} B_{x^*},$$

and hence

$$\begin{aligned}
(b) &\leq C2^{-k\varepsilon'} \sum_{j \geq k-5} 2^{-j(s-\varepsilon')} \left( \int_{17B(x, 2^{-k})} g_j^t d\mu \right)^{1/t} \\
&\leq C2^{-ks} \sum_{j \geq k-5} 2^{(k-j)(s-\varepsilon')} (\mathcal{M} g_j^t(x))^{1/t} \\
&\leq Cd(x, y)^s \sum_{j \geq k-5} 2^{(k-j)(s-\varepsilon')} (\mathcal{M} g_j^t(x))^{1/t}.
\end{aligned}$$

For the third term (c), we have, using similar estimates as above

$$\begin{aligned}
(c) &= |m_u(2^l B_{x^*} \cap S) - m_u(B_{x^*} \cap S)| \\
&\leq \sum_{i=0}^{l-1} |m_u(2^i B_{x^*} \cap S) - m_u(2^{i+1} B_{x^*} \cap S)| \\
&\leq C \sum_{i=0}^{l-1} 2^{-(m-i-1)\varepsilon'} \sum_{j \geq m-i-3} 2^{-j(s-\varepsilon')} \left( \int_{2^{i+2} B_{x^*}} g_j^t d\mu \right)^{1/t}.
\end{aligned}$$

Since  $B_x \subset B_{x^*} \subset 5B_x$ , estimating the sum by the  $(l-1)$ . term and using (5.5), we have

$$\begin{aligned}
(c) &\leq C \sum_{i=0}^{l-1} 2^{-(m-i-1)s} \sum_{j \geq m-i-3} 2^{(m-i-1-j)(s-\varepsilon')} (\mathcal{M} g_j^t(x))^{1/t} \\
&\leq C2^{-ks} \sum_{j \geq k-5} 2^{(k-j)(s-\varepsilon')} (\mathcal{M} g_j^t(x))^{1/t} \\
&\leq Cd(x, y)^s \sum_{j \geq k-5} 2^{(k-j)(s-\varepsilon')} (\mathcal{M} g_j^t(x))^{1/t}.
\end{aligned}$$

CASE 3:  $x, y \in V \setminus S$ ,  $d(x, y) \geq \min\{\text{dist}(x, S), \text{dist}(y, S)\}$ .

We begin with inequality

$$\begin{aligned}
|\tilde{E}u(x) - \tilde{E}u(y)| &\leq |\tilde{E}u(x) - m_u(B_{x^*} \cap S)| + |\tilde{E}u(y) - m_u(B_{y^*} \cap S)| \\
&\quad + |m_u(B_{x^*} \cap S) - m_u(B_{y^*} \cap S)| \\
&= (1) + (2) + (3),
\end{aligned}$$

where  $y^* \in B_y \cap S$  and  $B_{y^*}$  are chosen similarly as point  $x^*$  and ball  $B_{x^*}$  for  $x$  in the beginning of case 2. The radii of balls  $B_{x^*}$  and  $B_{y^*}$  are denoted by  $2^{-m_x}$  and  $2^{-m_y}$ .

We may assume that  $\text{dist}(x, S) \leq \text{dist}(y, S)$ . Then

$$r(y) = \frac{1}{10} \text{dist}(y, S) \leq \frac{1}{10}(d(x, y) + \text{dist}(x, S)) \leq \frac{1}{5}d(x, y),$$

and hence  $\text{dist}(y, S) \leq 2d(x, y)$ ,  $d(x, x^*) < 2^{-k+1}$  and  $d(y, y^*) < 3 \cdot 2^{-k}$ . Hence estimates for (1) and (2) follow similarly as for the first term of (5.4).



For the last term (3), let  $K \geq 0$  be the smallest integer such that the radius of the ball  $2^K B_{y^*}$  is at least  $2^{-k}$ , that is,  $2^{-k} \leq 2^{K-m_y} < 2^{-k+1}$ . Then

$$\begin{aligned} & |m_u(B_{x^*} \cap S) - m_u(B_{y^*} \cap S)| \\ & \leq |m_u(B_{y^*} \cap S) - m_u(2^K B_{y^*} \cap S)| + |m_u(2^K B_{y^*} \cap S) - m_u(B_{x^*} \cap S)| \\ & = (\alpha) + (\beta). \end{aligned}$$

We begin with  $(\alpha)$ . If  $K = 0$ , then  $(\alpha) = 0$ . If  $K > 0$ , then the radius of  $2^{K-1} B_{y^*}$  is at most  $2^{-k}$ . This together with the fact that  $d(y, y^*) < 3 \cdot 2^{-k}$  implies that

$$2^{K+1} B_{y^*} \subset B(y, 7 \cdot 2^{-k}) \subset 5 \cdot 2^{K+1} B_{y^*}.$$

Hence, using (3.9), we have

$$\begin{aligned} (\alpha) & \leq \sum_{i=0}^{K-1} |m_u(2^i B_{y^*} \cap S) - m_u(2^{i+1} B_{y^*} \cap S)| \\ & \leq C \sum_{i=0}^{K-1} 2^{-(m_y-i-1)\varepsilon'} \sum_{j \geq m_y-i-3} 2^{-j(s-\varepsilon')} \left( \int_{2^{i+2} B_{y^*}} g_j^t d\mu \right)^{1/t} \\ & \leq C \sum_{i=0}^{K-1} 2^{-(m_y-i-1)\varepsilon'} \sum_{j \geq m_y-i-3} 2^{-j(s-\varepsilon')} (\mathcal{M} g_j^t(y))^{1/t}. \end{aligned}$$

As in the case 2 (c), we estimate the sum by the  $(K-1)$ . term and use the fact that

$$(5.6) \quad 2^{K-1-m_y} < 2^{-k} \leq 2^{K-m_y}$$

and obtain

$$(\alpha) \leq C d(x, y)^s \sum_{j \geq k-1} 2^{(k-j)(s-\varepsilon')} (\mathcal{M} g_j^t(y))^{1/t}.$$

For  $(\beta)$ , let  $L \geq 0$  be the smallest integer such that  $2^K B_{y^*} \subset 2^L B_{x^*}$ . Now, by the selection of  $L$  and (5.6),

$$(5.7) \quad 2^{-k} \leq 2^{L-m_x} < 2^{-k+4},$$

and hence  $2^L B_{x^*} \subset 22 \cdot 2^K B_{y^*}$ ,  $2^L B_{x^*} \subset B(x, 2^{-k})$  and  $B(x, 2^{-k}) \subset 3 \cdot 2^L B_{x^*}$ . Now

$$\begin{aligned} (\beta) & = |m_u(2^K B_{y^*} \cap S) - m_u(B_{x^*} \cap S)| \\ & \leq |m_u(2^K B_{y^*} \cap S) - m_u(2^L B_{x^*} \cap S)| + |m_u(2^L B_{x^*} \cap S) - m_u(B_{x^*} \cap S)|, \end{aligned}$$

where, by similar estimates as above and (5.7),

$$|m_u(2^K B_{y^*} \cap S) - m_u(2^L B_{x^*} \cap S)| \leq C d(x, y)^s \sum_{j \geq k-6} 2^{(k-j)(s-\varepsilon')} (\mathcal{M} g_j^t(x))^{1/t}.$$

Similarly as for  $(\alpha)$  above, we obtain

$$|m_u(2^L B_{x^*} \cap S) - m_u(B_{x^*} \cap S)| \leq C d(x, y)^s \sum_{j \geq k-6} 2^{(k-j)(s-\varepsilon')} (\mathcal{M} g_j^t(x))^{1/t}.$$

CASE 4:  $x, y \in V \setminus S$ ,  $d(x, y) < \min\{\text{dist}(x, S), \text{dist}(y, S)\}$ .

We may assume that  $\text{dist}(x, S) \leq \text{dist}(y, S)$ . By the properties of the functions  $\varphi_i$  and the fact that  $B_i \subset B_{x^*}$  with comparable measures whenever  $i \in I_x \cup I_y$ , we can use similar estimates as for the first term of (5.4) and obtain

$$(5.8) \quad \begin{aligned} |\tilde{E}u(x) - \tilde{E}u(y)| &= \left| \sum_{i \in I_x \cup I_y} (\varphi_i(x) - \varphi_i(y)) (m_u(B_i^* \cap S) - m_u(B_{x^*} \cap S)) \right| \\ &\leq C \frac{d(x, y)}{r(x)} 2^{-m_x \varepsilon'} \sum_{j \geq m_x - 2} 2^{-j(s - \varepsilon')} (\mathcal{M} g_j^t(x))^{1/t}. \end{aligned}$$

Using the assumptions  $0 < \delta < 1 - s$ ,  $r(x) < 2^{-m_x}$ ,  $d(x, y) < \text{dist}(x, S) = 10r(x)$  and  $d(x, y) < 2^{-k}$ , we have

$$\begin{aligned} d(x, y) r(x)^{-1} 2^{-m_x \varepsilon'} &\leq C d(x, y) r(x)^{s + \delta - 1} 2^{m_x(s - \varepsilon' + \delta)} \\ &\leq C d(x, y)^{s + \delta} 2^{m_x(s - \varepsilon' + \delta)} \\ &\leq C d(x, y)^s 2^{(m_x - k)\delta + m_x(s - \varepsilon')}. \end{aligned}$$

This together with (5.8) implies that

$$|\tilde{E}u(x) - \tilde{E}u(y)| \leq C d(x, y)^s \sum_{j = m_x - 2}^{\infty} 2^{(m_x - k)\delta + (m_x - j)(s - \varepsilon')} (\mathcal{M} g_j^t(x))^{1/t}.$$

By splitting the sum in two parts and using the facts  $m_x \leq j + 2$  and  $m_x \leq k$ , we obtain

$$\begin{aligned} &\sum_{j = m_x - 2}^{\infty} 2^{(m_x - k)\delta + (m_x - j)(s - \varepsilon')} (\mathcal{M} g_j^t(x))^{1/t} \\ &= \sum_{j = m_x - 2}^{k-1} 2^{(m_x - k)\delta + (m_x - j)(s - \varepsilon')} (\mathcal{M} g_j^t(x))^{1/t} \\ &\quad + \sum_{j = k}^{\infty} 2^{(m_x - k)\delta + (m_x - j)(s - \varepsilon')} (\mathcal{M} g_j^t(x))^{1/t} \\ &\leq C \left( \sum_{j = -\infty}^{k-1} 2^{(j - k)\delta} (\mathcal{M} g_j^t(x))^{1/t} + \sum_{j = k}^{\infty} 2^{(k - j)(s - \varepsilon')} (\mathcal{M} g_j^t(x))^{1/t} \right), \end{aligned}$$

which implies the claim in case 4. Cases 1-4 show that  $(C\tilde{g}_k)$  is a fractional  $s$ -gradient for the extension  $\tilde{E}u$ .  $\square$

Next will estimate the norm of the fractional  $s$ -gradient of the local extension. Recall from (5.3) that

$$\tilde{g}_k(x) = \sum_{j = -\infty}^{k-1} 2^{(j - k)\delta} (\mathcal{M} g_j^t(x))^{1/t} + \sum_{j = k-6}^{\infty} 2^{(k - j)(s - \varepsilon')} (\mathcal{M} g_j^t(x))^{1/t},$$

where  $0 < \delta < 1 - s$ ,  $0 < \varepsilon' < s$  and  $0 < t < \min\{p, q\}$ .

**Lemma 5.3.**  $\|(\tilde{g}_k)\|_{L^p(V, l^q)} \leq C \|(g_k)\|_{L^p(S, l^q)}.$

*Proof.* We estimate only the  $L^p(V, l^q)$  norm of

$$\left( \sum_{j=-\infty}^{k-1} 2^{(j-k)\delta} (\mathcal{M} g_j^t(x))^{1/t} \right)_{k \in \mathbb{Z}},$$

since another part can be estimated in a similar way.

Lemma 3.1 implies that

$$\sum_{k \in \mathbb{Z}} \left( \sum_{j=-\infty}^{k-1} 2^{(j-k)\delta} (\mathcal{M} g_j^t(x))^{1/t} \right)^q \leq C \sum_{j \in \mathbb{Z}} (\mathcal{M} g_j^t(x))^{q/t}$$

Hence, using a version of the Fefferman–Stein vector valued maximal function theorem for metric space with a doubling measure, proved in [35, Thm 1.3], [10, Thm 1.2] (for the original version, see [8, Thm 1]), we obtain

$$\begin{aligned} \|(\tilde{g}_k)_{k \in \mathbb{Z}}\|_{L^p(V, l^q)} &\leq \|(M g_k^t)_{k \in \mathbb{Z}}\|_{L^{p/t}(V, l^{q/t})}^{1/t} \leq C \|(g_k^t)_{k \in \mathbb{Z}}\|_{L^{p/t}(X, l^{q/t})}^{1/t} \\ &= C \|\vec{g}\|_{L^p(X, l^q)} = C \|\vec{g}\|_{L^p(S, l^q)}, \end{aligned}$$

where the last equality holds, since  $\vec{g} \equiv 0$  outside  $S$ .  $\square$

If  $u \in N_{p,q}^s(S)$  and  $(g_k)_{k \in \mathbb{Z}} \in \mathbb{D}^s(u)$  with

$$\|(g_k)\|_{l^q(L^p(S))} < 2 \inf_{(h_k) \in \mathbb{D}^s(u)} \|(h_k)\|_{l^q(L^p(S))},$$

then we proceed as in the Triebel–Lizorkin case. Lemma 5.2 gives a fractional  $s$ -gradient  $(\tilde{g}_k)_{k \in \mathbb{Z}}$  for the local extension, and the norm estimate corresponding Lemma 5.3 follows from the lemma below.

**Lemma 5.4.**  $\|(\tilde{g}_k)\|_{l^q(L^p(V))} \leq C \|(g_k)\|_{l^q(L^p(S))}$ .

*Proof.* As in the proof of Lemma 5.3, we estimate the first part of  $(\tilde{g}_k)$  only. The second part can be estimated similarly.

Assume first that  $p \geq 1$ . By the Minkowski inequality and the boundedness of the Hardy–Littlewood maximal operator in  $L^r$ ,  $r > 1$ , we have

$$\begin{aligned} \left\| \sum_{j=-\infty}^k 2^{(j-k)\delta} (\mathcal{M} g_j^t)^{1/t} \right\|_{L^p(V)} &\leq \sum_{j=-\infty}^k 2^{(j-k)\delta} \|(\mathcal{M} g_j^t)^{1/t}\|_{L^p(V)} \\ &\leq \sum_{j=-\infty}^k 2^{(j-k)\delta} \|g_j\|_{L^p(X)}, \end{aligned}$$

and Lemma 3.1 implies that

$$\sum_{k \in \mathbb{Z}} \left( \sum_{j=-\infty}^k 2^{(j-k)\delta} \|g_j\|_{L^p(V)} \right)^q \leq C \sum_{j \in \mathbb{Z}} \|g_j\|_{L^p(X)}^q.$$

Assume then that  $0 < p < 1$ . Using inequality (3.1) and the boundedness of the Hardy–Littlewood maximal operator in  $L^r$ ,  $r > 1$ , we obtain

$$\begin{aligned} \left\| \sum_{j=-\infty}^k 2^{(j-k)\delta} (\mathcal{M} g_j^t)^{1/t} \right\|_{L^p(V)}^p &\leq \sum_{j=-\infty}^k 2^{(j-k)\delta p} \| (\mathcal{M} g_j^t)^{1/t} \|_{L^p(V)}^p \\ &\leq \sum_{j=-\infty}^k 2^{(j-k)\delta p} \| g_j \|_{L^p(X)}^p. \end{aligned}$$

Hence, using Lemma 3.1, we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left\| \sum_{j=-\infty}^k 2^{(j-k)\delta} (\mathcal{M} g_j^t)^{1/t} \right\|_{L^p(V)}^q &\leq \sum_{k \in \mathbb{Z}} \left( \sum_{j=-\infty}^k 2^{(j-k)\delta p} \| g_j \|_{L^p(X)}^p \right)^{q/p} \\ &\leq C \sum_{j \in \mathbb{Z}} \| g_j \|_{L^p(X)}^q. \end{aligned}$$

The desired norm estimate follows in both cases because  $\vec{g} \equiv 0$  outside  $S$ .  $\square$

### The final extension:

Now we are ready to define the final extension. Let  $\Psi: X \rightarrow [0, 1]$  be an  $L$ -Lipschitz cut-off function such that  $\Psi|_S = 1$  and  $\Psi|_{X \setminus V} = 0$ . We define an extension operator by setting

$$Eu = \Psi \tilde{E}u.$$

Then  $Eu = u$  in  $S$  and, by (5.2),

$$\|Eu\|_{L^p(X)} \leq \|\tilde{E}u\|_{L^p(V)} \leq C\|u\|_{L^p(S)}.$$

In the Triebel–Lizorkin case, (5.2) together with Lemmas 5.2 and 5.3 imply that  $\tilde{E}u \in M_{p,q}^s(V)$  and  $\|(\tilde{g}_k)\|_{L^p(V, l^q)} \leq \|(g_k)\|_{L^p(S, l^q)}$ . Now, by Lemma 3.10, the sequence  $(g'_k)_{k \in \mathbb{Z}}$ ,

$$g'_k = \begin{cases} (\tilde{g}_k + 2^{sk+2} |\tilde{E}u|) \chi_{\text{supp } \Psi}, & \text{if } k < k_L, \\ (\tilde{g}_k + 2^{k(s-1)} L |\tilde{E}u|) \chi_{\text{supp } \Psi}, & \text{if } k \geq k_L, \end{cases}$$

where  $k_L$  is the integer such that  $2^{k_L-1} < L \leq 2^{k_L}$ , is a fractional  $s$ -gradient of  $Eu$  and it satisfies norm estimate

$$\|\vec{g}'\|_{L^p(X, l^q)} \leq C\|\tilde{E}u\|_{M_{p,q}^s(V)} \leq C\|u\|_{M_{p,q}^s(S)}.$$

Hence  $Eu \in M_{p,q}^s(X)$  and  $\|Eu\|_{M_{p,q}^s(X)} \leq C\|u\|_{M_{p,q}^s(S)}$ . The Besov case follows similarly by using Lemma 5.3 instead of Lemma 5.3 and Remark 3.11.

This concludes the proof of Theorem 1.2.  $\square$

## 6. MEASURE DENSITY FROM EXTENSION

The main theorem of this section shows that if the space  $X$  is  $Q$ -regular and geodesic, then the measure density of a domain is a necessary condition for the extension property of functions from Hajlasz–Besov and Hajlasz–Triebel–Lizorkin spaces. The analogous result for functions from Hajlasz–Sobolev spaces was proved in [15, Thm 5], and the proof given below is inspired by the corresponding proof in that paper. The main tools

in the proof are Lipschitz estimates from Section 3.3 and embedding theorems, both the old ones and Theorem 4.4. The assumption that  $X$  is geodesic is used only to get the property  $\mu(\partial B) = 0$  for all balls  $B$ .

**Theorem 6.1.** *Let  $X$  be a  $Q$ -regular, geodesic metric measure space. Let  $0 < s < 1$ ,  $0 < p < \infty$ , and  $0 < q \leq \infty$ . If  $\Omega \subset X$  is an  $M_{p,q}^s$ -extension domain (or an  $N_{p,q}^s$ -extension domain), then it satisfies measure density condition (1.1).*

*Proof.* First we assume that  $\Omega$  is an  $M_{p,q}^s$ -extension domain for some  $0 < s < 1$ ,  $0 < p < \infty$ , and  $0 < q \leq \infty$ . To show that the measure density condition holds, let  $x \in \overline{\Omega}$  and  $0 < r \leq 1$ , and let  $B = B(x, r)$ . We may assume that  $\Omega \setminus B(x, r) \neq \emptyset$ , otherwise the measure density condition is obviously satisfied.

We split the proof into three different cases depending on the size of  $sp$ .

CASE 1:  $0 < sp < Q$ . By the proof of [15, Proposition 13], the geodesicity of  $X$  implies that  $\mu(\partial B(x, R)) = 0$  for every  $R > 0$ . Hence there exist radii  $0 < \tilde{r} < \tilde{\tilde{r}} < r$  such that

$$\mu(B(x, \tilde{\tilde{r}}) \cap \Omega) = \frac{1}{2}\mu(B(x, \tilde{r}) \cap \Omega) = \frac{1}{4}\mu(B(x, r) \cap \Omega).$$

Let  $u: \Omega \rightarrow [0, 1]$ ,

$$u(y) = \begin{cases} 1, & \text{if } y \in B(x, \tilde{\tilde{r}}) \cap \Omega, \\ \frac{\tilde{r}-d(x,y)}{\tilde{r}-\tilde{\tilde{r}}}, & \text{if } y \in B(x, \tilde{r}) \setminus B(x, \tilde{\tilde{r}}) \cap \Omega, \\ 0, & \text{if } y \in \Omega \setminus B(x, \tilde{r}). \end{cases}$$

Since the function  $u$  is  $1/(\tilde{r} - \tilde{\tilde{r}})$ -Lipschitz and  $\|u\|_\infty \leq 1$ , by Corollary 3.12 and the fact that  $0 < \tilde{r} - \tilde{\tilde{r}} < 1$ , we have

$$(6.1) \quad \|u\|_{M_{p,q}^s(\Omega)} \leq C\mu(B(x, \tilde{r}) \cap \Omega)^{1/p}(\tilde{r} - \tilde{\tilde{r}})^{-s}.$$

We want to find a good lower bound for  $\|u\|_{M_{p,q}^s(\Omega)}$ . Let  $v \in M_{p,q}^s(X)$  be an extension of  $u$ , and let  $(h_k)_{k \in \mathbb{Z}} \in \mathbb{D}^s(v)$  be such that

$$\|(h_k)_{k \in \mathbb{Z}}\|_{L^p(X, l^q)} \leq C\|v\|_{M_{p,q}^s(X)}.$$

Since

$$|v(z) - v(y)| \leq Cd(z, y)^s \left( \sup_{k \in \mathbb{Z}} h_k(z) + \sup_{k \in \mathbb{Z}} h_k(y) \right)$$

for almost every  $z, y \in X$ , we have that  $v \in M^{s,p}(X)$  with an  $s$ -gradient  $H = \sup_{k \in \mathbb{Z}} h_k$ .

Now, by Lemma 3.4,

$$(6.2) \quad \inf_{c \in \mathbb{R}} \left( \int_{B(x,1)} |v - c|^{p^*(s)} d\mu \right)^{1/p^*(s)} \leq C \left( \int_{B(x,2)} H^p d\mu \right)^{1/p},$$

where  $p^*(s) = Qp/(Q - ps)$  and, by the selection of  $(h_k)_{k \in \mathbb{Z}}$ ,

$$\left( \int_{B(x,2)} H^p d\mu \right)^{1/p} \leq \left( \int_X \sup_{k \in \mathbb{Z}} h_k^p d\mu \right)^{1/p} \leq C\|v\|_{M_{p,q}^s(X)}.$$

Inequality (6.2), together with the last estimate, the  $Q$ -regularity and the fact that  $v$  is an extension of  $u$ , implies the existence of  $c_0$  such that

$$\left( \int_{B(x,1)} |v - c_0|^{p^*(s)} d\mu \right)^{1/p^*(s)} \leq C\|v\|_{M_{p,q}^s(X)} \leq C\|u\|_{M_{p,q}^s(\Omega)}.$$

Hence, by the Chebyshev inequality, we have

$$(6.3) \quad (\mu(\{y \in B(x, 1) : |v(y) - c_0| > \lambda\}))^{1/p^*(s)} \leq \frac{C}{\lambda} \|u\|_{M_{p,q}^s(\Omega)}.$$

Since  $u = v = 1$  on  $B(x, \tilde{r}) \cap \Omega$  and  $u = v = 0$  on  $(B(x, r) \setminus B(x, \tilde{r})) \cap \Omega$ , we have that

$$|v - c_0| \geq 1/2$$

on at least one of the sets  $B(x, \tilde{r}) \cap \Omega$  and  $(B(x, r) \setminus B(x, \tilde{r})) \cap \Omega$ . Since the two sets have measures comparable to the measure of  $B(x, \tilde{r}) \cap \Omega$ , (6.3) with  $\lambda = 1/2$  gives

$$\mu(B(x, \tilde{r}) \cap \Omega)^{1/p^*(s)} \leq C \|u\|_{M_{p,q}^s(\Omega)}.$$

This together with (6.1) shows that

$$\mu(B(x, \tilde{r}) \cap \Omega)^{1/p^*(s)} \leq C(\tilde{r} - \tilde{r})^{-s} \mu(B(x, \tilde{r}) \cap \Omega)^{1/p},$$

and hence

$$(6.4) \quad \tilde{r} - \tilde{r} \leq C \mu(B(x, \tilde{r}) \cap \Omega)^{1/Q}.$$

Now, defining radii  $r_j$ ,  $j = 0, 1, \dots$ , as

$$r_0 = r, \quad r_{j+1} = \tilde{r}_j,$$

we have

$$\mu(B(x, r_j) \cap \Omega) = 2^{-j} \mu(B(x, r) \cap \Omega),$$

which implies that  $r_j \rightarrow 0$  as  $j \rightarrow \infty$ . Applying inequality (6.4) for  $r_{j+1}$ , we obtain

$$r_{j+1} - r_{j+2} \leq C \mu(B(x, r_{j+1}) \cap \Omega)^{1/Q} \leq C 2^{-j/Q} \mu(B(x, r) \cap \Omega)^{1/Q},$$

and hence

$$\tilde{r} = r_1 = \sum_{j=0}^{\infty} (r_{j+1} - r_{j+2}) \leq C \mu(B(x, r) \cap \Omega)^{1/Q}.$$

Since it was proved in [15, Lemma 14], that if measure density condition (1.1) holds for all  $x \in \overline{\Omega}$  and all  $0 < r \leq 1$  such that  $r \leq 10\tilde{r}$ , it holds for all  $x \in \overline{\Omega}$  and all  $0 < r \leq 1$ , we are done in this case. Note that the assumption of connectedness of  $\Omega$  is essential in the cited lemma.

CASE 2:  $sp > Q$ . Let  $u: \Omega \rightarrow [0, 1]$ ,

$$u(y) = \begin{cases} 1 - \frac{d(x,y)}{r}, & \text{if } y \in B(x, r), \\ 0, & \text{if } y \in \Omega \setminus B(x, r). \end{cases}$$

Since the function  $u$  is  $r^{-1}$ -Lipschitz and  $\|u\|_{\infty} \leq 1$ , using Corollary 3.12 and the fact that  $0 < r < 1$ , we obtain

$$(6.5) \quad \|u\|_{M_{p,q}^s(\Omega)} \leq C (\mu(B(x, r) \cap \Omega))^{1/p} r^{-s}.$$

Let  $v \in M_{p,q}^s(X)$  be an extension of  $u$ , and let  $(h_k)_{k \in \mathbb{Z}} \in \mathbb{D}^s(v)$  be such that

$$(6.6) \quad \|(h_k)_{k \in \mathbb{Z}}\|_{L^p(X, l^q)} \leq C \|v\|_{M_{p,q}^s(X)} \leq C \|u\|_{M_{p,q}^s(\Omega)}.$$



As in the case  $sp < Q$ , since  $v \in M_{p,q}^s(X)$  and  $(h_k)_{k \in \mathbb{Z}}$  is its fractional  $s$ -gradient, we have that

$$|v(z) - v(y)| \leq Cd(z, y)^s \left( \sup_{k \in \mathbb{Z}} h_k(z) + \sup_{k \in \mathbb{Z}} h_k(y) \right)$$

for almost every  $z, y \in X$ . Thus  $v \in M^{s,p}(X)$  and  $H = \sup_{k \in \mathbb{Z}} h_k$  is its  $s$ -gradient. By the analogue of [13, Lemma 8] (the proof goes in the same way),

$$(6.7) \quad |v(x) - v(y)| \leq Cr^{Q/p} d(x, y)^{s-Q/p} \left( \int_{B(x, 5r)} H^p d\mu \right)^{1/p}.$$

Since  $v(x) = u(x) = 1$  and  $v(y) = u(y) = 0$  for some  $y \in (\Omega \setminus B(x, r)) \cap B(x, 2r)$  (we can assume that (6.7) holds for these particular points  $x$  and  $y$ ), using (6.6), (6.5) and the  $Q$ -regularity, we obtain

$$\begin{aligned} 1 &\leq Cr^{Q/p} r^{s-Q/p} \left( \int_{B(x, 5r)} H^p d\mu \right)^{1/p} \leq Cr^{s-Q/p} \|(h_k)_{k \in \mathbb{Z}}\|_{L^p(X, l^q)} \\ &\leq Cr^{-Q/p} \mu(B(x, r) \cap \Omega)^{1/p}, \end{aligned}$$

which implies the measure density by the  $Q$ -regularity.

CASE 3:  $sp = Q$ . We will use the following modification of [18, Thm 5.9]. Below,  $\mathcal{H}_\infty^1$  is the Hausdorff content of dimension 1.

**Lemma 6.2.** *Let  $X$  be a  $Q$ -regular space,  $Q \geq 1$ . Let  $E$  and  $F$  be disjoint subsets of a ball  $B = B(x, r)$  such that*

$$(6.8) \quad \min\{\mathcal{H}_\infty^1(E), \mathcal{H}_\infty^1(F)\} \geq \lambda r,$$

for some  $0 < \lambda \leq 1$ . Then there is a constant  $C \geq 1$ , depending only on  $X$ , such that

$$\int_{20B} g^p d\mu \geq C\lambda$$

whenever  $u$  is locally integrable,  $g$  is a  $(Q/p)$ -gradient of  $u$  in  $20B$ , every point in  $E \cup F$  is a Lebesgue point of  $u$ ,  $u|_E \geq 1$  and  $u|_F \leq 0$ .

Let  $B = B(x, r)$  and let  $A = \frac{2}{3}B \setminus \frac{1}{3}B$ . Let  $u: \Omega \rightarrow [0, 1]$ ,

$$u(y) = \begin{cases} 1, & \text{if } y \in \frac{1}{3}B \cap \Omega, \\ 2 - \frac{3d(x, y)}{r}, & \text{if } y \in A \cap \Omega, \\ 0, & \text{if } y \in \Omega \setminus \frac{2}{3}B. \end{cases}$$

The function  $u$  is  $3/r$ -Lipschitz and, as above, by Corollary 3.12, we obtain

$$\|u\|_{M_{p,q}^s(\Omega)} \leq C(\mu(B(x, r) \cap \Omega))^{1/p} r^{-s}.$$

Let  $v \in M_{p,q}^s(X)$  be an extension of  $u$  and let  $(h_k)_{k \in \mathbb{Z}} \in \mathbb{D}^s(v)$  such that

$$\|(h_k)_{k \in \mathbb{Z}}\|_{L^p(X, l^q)} \leq C\|v\|_{M_{p,q}^s(X)} \leq C\|u\|_{M_{p,q}^s(\Omega)}.$$

Using the connectivity of  $\Omega$  and the fact that the 1-Lipschitz function  $y \mapsto d(x, y)$  does not increase the Hausdorff 1-content, we obtain, as in [15, p.665], that

$$\min\left\{\mathcal{H}_\infty^1\left(\frac{1}{3}B \cap \Omega\right), \mathcal{H}_\infty^1\left(B \setminus \frac{2}{3}B \cap \Omega\right)\right\} \geq \frac{r}{3}.$$

Applying Lemma 6.2 to the function  $v$  with a  $(Q/p)$ -gradient  $H = \sup_{k \in \mathbb{Z}} h_k$ , we obtain

$$C \leq \int_{20B} H^p d\mu \leq \mu(B \cap \Omega) r^{-Q},$$

which implies the measure density by the  $Q$ -regularity.

We have shown that the extension property for Triebel–Lizorkin spaces implies the measure density condition for a domain.

In order to obtain the analogous result for Besov spaces, we have to make the following modifications in the proof given above. Observe that in all three cases, the  $M_{p,q}^s(\Omega)$ -norms and  $N_{p,q}^s(\Omega)$ -norms of the chosen test functions have the same upper bounds, see Lemma 3.12.

CASE 1:  $0 < sp < Q$ . Instead of (6.3), we use an estimate

$$(\mu(\{y \in X : |v(y) - c_0| > \lambda\}))^{1/p^*(s)} \leq \frac{C}{\lambda} \|u\|_{N_{p,q}^s(X)},$$

which follows from the case  $q = \infty$  of Theorem 4.4 and the fact that  $\|u\|_{N_{p,\infty}^s(X)} \leq \|u\|_{N_{p,q}^s(X)}$  for  $0 < q \leq \infty$ .

CASE 2:  $sp > Q$ . Instead of (6.7), we use the following lemma.

**Lemma 6.3.** *Let  $X$  be a  $Q$ -regular space,  $Q \geq 1$ . Let  $0 < s < 1$  and  $sp > Q$ . There is a constant  $C > 0$ , such that for each  $u \in \dot{N}_{p,q}^s(X)$ ,*

$$|u(x) - u(y)| \leq Cd(x, y)^{s-Q/p} \|u\|_{\dot{N}_{p,q}^s(X)}$$

for almost every  $x, y \in X$ .

*Proof.* Using Poincaré inequality (3.2), the Hölder inequality and the  $Q$ -regularity, we obtain

$$\int_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq Cr^{s-Q/p} \|u\|_{\dot{N}_{p,q}^s(X)},$$

for every  $x \in X$  and  $r > 0$ . The claim follows now from [29, Thm 4].  $\square$

CASE 3:  $sp = Q$ . Let the radii  $\tilde{r}$  and  $\tilde{r}$  and the function  $u$  be as in Case 1. Let  $v \in N_{p,q}^s(X)$  be an extension of  $u$  with  $\|v\|_{N_{p,q}^s(X)} \leq C\|u\|_{N_{p,q}^s(\Omega)}$ . By Poincaré inequality (3.2), the Hölder inequality and the  $Q$ -regularity,  $v \in \text{BMO}(X)$  and

$$\|v\|_{\text{BMO}(X)} \leq C\|v\|_{N_{p,q}^s(X)}.$$

Hence, by the John–Nirenberg theorem [4, Thm 2.2],

$$(6.9) \quad \inf_{b \in \mathbb{R}} \int_{B(x,r)} \exp\left(\frac{|v - b|}{C\|v\|_{N_{p,q}^s(X)}}\right) d\mu \leq C.$$

By Corollary 3.12, we have

$$(6.10) \quad \|v\|_{N_{p,q}^s(X)} \leq C\|u\|_{N_{p,q}^s(\Omega)} \leq C(\tilde{r} - \tilde{r})^{-s} \mu(B(x, \tilde{r}) \cap \Omega).$$

Since  $u = v = 1$  on  $B(x, \tilde{r}) \cap \Omega$  and  $u = v = 0$  on  $(B(x, r) \setminus B(x, \tilde{r})) \cap \Omega$ , we have that  $|v - c_0| \geq 1/2$  on at least one of the sets  $B(x, \tilde{r}) \cap \Omega$  and  $(B(x, r) \setminus B(x, \tilde{r})) \cap \Omega$ .

Since the measures of these two sets are comparable to the measure of  $B(x, \tilde{r}) \cap \Omega$ , (6.9) and (6.10) imply

$$\mu(B(x, \tilde{r}) \cap \Omega) \exp(C^{-1}(\tilde{r} - \tilde{\tilde{r}})^s \mu(B(x, \tilde{r}) \cap \Omega)^{-1/p}) \leq Cr^Q,$$

which can be written in the form

$$\tilde{r} - \tilde{\tilde{r}} \leq C\mu(B(x, \tilde{r}) \cap \Omega)^{1/Q} \log \left( \frac{Cr^Q}{\mu(B(x, \tilde{r}) \cap \Omega)} \right)^{1/s}.$$

Now, defining radii  $r_j$ ,  $j = 0, 1, \dots$ , as

$$r_0 = r, \quad r_{j+1} = \tilde{r}_j,$$

we have

$$\mu(B(x, r_{j+1}) \cap \Omega) = 2^{-j} \mu(B(x, \tilde{r}) \cap \Omega),$$

which implies

$$\tilde{r} = r_1 = \sum_{j=0}^{\infty} (r_{i+1} - r_{i+2}) \leq C\mu(B(x, \tilde{r}) \cap \Omega)^{1/Q} \sum_{j=0}^{\infty} 2^{-j/Q} \log \left( \frac{C2^j r_j^Q}{\mu(B(x, \tilde{r}) \cap \Omega)} \right)^{1/s}.$$

A similar argument as in [14, p. 1228–1229], shows that

$$\sum_{j=0}^{\infty} 2^{-j/Q} \log \left( \frac{C2^j r_j^Q}{\mu(B(x, \tilde{r}) \cap \Omega)} \right)^{1/s} \leq C.$$

Thus, the measure density condition holds for all  $x \in \overline{\Omega}$  and  $0 < r \leq 1$  such that  $r \leq 10\tilde{r}$ , and the claim follows by [15, Lemma 14], which tells that it suffices to prove (1.1) in that case.  $\square$

**Remark 6.4.** The proof of Case 3 for Besov spaces is a modification of the proof of [14, Thm 1.b)]; if  $q < \infty$ , one could simplify the reasoning using the proof of Case 3 for Triebel–Lizorkin spaces with Lemma 6.2 replaced by [26, Lemma 3.3].

**Remark 6.5.** Since Hajlasz–Besov and Hajlasz–Triebel–Lizorkin functions do not see the sets of measure zero, it is also natural to discuss a connection between the extension property and an ”almost everywhere” variant of the measure density condition. Indeed, the proof of Theorem 6.1 shows that if  $S \subset X$  is connected and there exists a bounded extension operator  $E: N_{p,q}^s(S) \rightarrow N_{p,q}^s(X)$  (or  $E: M_{p,q}^s(S) \rightarrow M_{p,q}^s(X)$ ), then the set

$$\tilde{S} = \{x \in X : \mu(B(x, r) \cap S) > 0 \text{ for every } r > 0\}$$

satisfies the measure density condition. Since  $\mu(S \setminus \tilde{S}) = 0$ , it follows that, for a connected set  $S$ , a bounded extension operator exists if and only if (1.1) holds for almost every  $x \in S$  and every  $0 < r \leq 1$ .

## 7. EXTENSION THEOREMS FOR BESOV AND TRIEBEL–LIZORKIN SPACES IN $\mathbb{R}^n$

In this section we apply our general results to obtain extension results for classical Besov and Triebel–Lizorkin spaces defined in the Euclidean space.

**7.1. Besov spaces on subsets of  $\mathbb{R}^n$ .** Let  $S \subset \mathbb{R}^n$  be a measurable set and let  $t > 0$ . For  $h \in \mathbb{R}^n$ , define  $S - h = \{s - h : s \in S\}$  and  $S_h = S \cap (S - h)$ . We consider the following versions of the  $L^p$ -modulus of smoothness on  $S$ :

$$(7.1) \quad \omega(u, S, t)_p = \sup_{|h| \leq t} \left( \int_{S_h} |u(x+h) - u(x)|^p dx \right)^{1/p},$$

$$(7.2) \quad \begin{aligned} E_p(u, S, t) &= \left( \int_{B(0,t)} \int_{S_h} |u(x+h) - u(x)|^p dx dh \right)^{1/p} \\ &= \left( \int_S \frac{1}{|B(x,t)|} \int_{B(x,t) \cap S} |u(y) - u(x)|^p dy dx \right)^{1/p} \end{aligned}$$

and

$$(7.3) \quad \hat{E}_p(u, S, t) = \left( \int_S \frac{1}{|B(x,t)|} \inf_{c \in \mathbb{R}} \int_{B(x,t) \cap S} |u(y) - c|^p dy dx \right)^{1/p}.$$

Versions (7.1) and (7.2) were used, for example, in [7] and (7.3) in [38], [39]. Note that (7.2) and (7.3) are connected to the smoothness functions  $C_t^{s,r}u(x)$  and  $I_t^{s,r}u(x)$  from [11, Def. 1.1], The Besov spaces  $B_{p,q}^s(S)$ ,  $\mathcal{B}_{p,q}^s(S)$  and  $\hat{\mathcal{B}}_{p,q}^s(S)$  consist of measurable functions for which the norms

$$\begin{aligned} \|u\|_{B_{p,q}^s(S)} &= \|u\|_{L^p(S)} + \left( \int_0^1 (t^{-s} \omega_p(u, S, t))^q \frac{dt}{t} \right)^{1/q}, \\ \|u\|_{\mathcal{B}_{p,q}^s(S)} &= \|u\|_{L^p(S)} + \left( \int_0^1 (t^{-s} E_p(u, S, t))^q \frac{dt}{t} \right)^{1/q} \end{aligned}$$

and

$$\|u\|_{\hat{\mathcal{B}}_{p,q}^s(S)} = \|u\|_{L^p(S)} + \left( \int_0^1 (t^{-s} \hat{E}_p(u, S, t))^q \frac{dt}{t} \right)^{1/q}$$

are finite respectively.

The following theorem describes how these spaces are related to each other, and to the Hajlasz–Besov space  $N_{p,q}^s$ .

**Theorem 7.1.** *Let  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ . Then*

$$N_{p,q}^s(S) \subset B_{p,q}^s(S) \subset \mathcal{B}_{p,q}^s(S) \subset \hat{\mathcal{B}}_{p,q}^s(S),$$

*and there is a constant  $C > 0$  such that*

$$(7.4) \quad \|\cdot\|_{\hat{\mathcal{B}}_{p,q}^s(S)} \leq \|\cdot\|_{\mathcal{B}_{p,q}^s(S)} \leq \|\cdot\|_{B_{p,q}^s(S)} \leq C \|\cdot\|_{N_{p,q}^s(S)},$$

*for each measurable set  $S \subset \mathbb{R}^n$ .*

*If  $S$  satisfies measure density condition (1.1), then*

$$(7.5) \quad N_{p,q}^s(S) = B_{p,q}^s(S) = \mathcal{B}_{p,q}^s(S) = \hat{\mathcal{B}}_{p,q}^s(S)$$

*with equivalent norms.*

*Proof.* Let  $S \subset \mathbb{R}^n$  be a measurable set. The first two inequalities in (7.4) are obvious, since  $\hat{E}_p(u, S, t) \leq E_p(u, S, t) \leq \omega(u, S, t)_p$  for all  $t > 0$ . In order to show the last inequality, let  $u \in N_{p,q}^s(S)$ ,  $(g_k)_{k \in \mathbb{Z}} \in \mathbb{D}^s(u)$  and  $k \in \mathbb{Z}$ . Then

$$\begin{aligned} \omega(u, S, 2^{-k})_p &= \sup_{j \geq k} \sup_{2^{-j-1} \leq |h| < 2^{-j}} \left( \int_{S_h} |u(x+h) - u(x)|^p dx \right)^{1/p} \\ &\leq C \sup_{j \geq k} 2^{-js} \left( \int_{S_h} g_j(x+h)^p + g_j(x)^p dx \right)^{1/p} \\ &\leq C \sup_{j \geq k} 2^{-js} \|g_j\|_{L^p(S)} \leq C \sum_{j \geq k} 2^{-js} \|g_j\|_{L^p(S)}. \end{aligned}$$

If  $0 < q < \infty$ , then by the estimate above and by Lemma 3.1, we obtain

$$\begin{aligned} \int_0^1 (t^{-s} \omega(u, S, t)_p)^q \frac{dt}{t} &\leq C \sum_{k \geq 0} (2^{ks} \omega(u, S, 2^{-k})_p)^q \leq C \sum_{k \geq 0} \left( \sum_{j \geq k} 2^{(k-j)sp} \|g_j\|_{L^p(S)}^p \right)^{q/p} \\ &\leq C \sum_{j \in \mathbb{Z}} \|g_j\|_{L^p(S)}^q. \end{aligned}$$

In the case  $q = \infty$ , we have

$$\begin{aligned} \sup_{0 < t < 1} t^{-s} \omega(u, S, t)_p &\leq C \sup_{k \geq 0} 2^{ks} \omega(u, S, 2^{-k})_p \\ &\leq C \sup_{k \geq 0} \left( \sum_{j \geq k} 2^{(k-j)sp} \|g_j\|_{L^p(S)}^p \right)^{1/p} \\ &\leq C \sup_{j \in \mathbb{Z}} \|g_j\|_{L^p(S)}. \end{aligned}$$

The claim follows by taking the infimum over all  $(g_k)_{k \in \mathbb{Z}} \in \mathbb{D}^s(u)$ .

Next assume that  $S$  satisfies the measure density condition. Then  $(S, d, \mu)$ , where  $d$  and  $\mu$  are the restrictions of the Euclidean metric and the Lebesgue measure to  $S$ , satisfies the doubling property locally, that is, for a given  $R > 0$ , there exists a constant  $C = C(n, c_m, R)$  such that

$$\mu(B(x, 2r)) \leq C \mu(B(x, r))$$

for all  $x \in S$  and  $0 < r \leq R$ . Now, the inclusion  $\hat{\mathcal{B}}_{p,q}^s(S) \subset N_{p,q}^s(S)$  and, hence, (7.5) follows essentially from the proof of [11, Thm 2.1].  $\square$

Theorem 7.1 implies that if  $\Omega$  is an extension domain for one of the spaces  $B_{p,q}^s$ ,  $\mathcal{B}_{p,q}^s$ ,  $\hat{\mathcal{B}}_{p,q}^s$ , then it is an extension domain for  $N_{p,q}^s$ . By combining Theorems 7.1, 6.1 and 1.2, we obtain the first main result of this section.

**Theorem 7.2.** *Let  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ . Then  $\Omega \subset \mathbb{R}^n$  is an extension domain for  $B_{p,q}^s$  (resp. for  $\mathcal{B}_{p,q}^s$  or for  $\hat{\mathcal{B}}_{p,q}^s$ ) if and only if it satisfies measure density condition (1.1).*

**Remark 7.3.** The definition of the space  $B_{p,q}^s(S)$  depends on the translation structure of  $\mathbb{R}^n$ , but the definitions of  $\mathcal{B}_{p,q}^s(S)$  and  $\hat{\mathcal{B}}_{p,q}^s(S)$  can be naturally extended to the metric

setting. With minor modifications in the proofs, we obtain the following counterparts of Theorems 7.1 and 7.2. We leave the details to the reader.

**Theorem 7.4.** *Let  $X$  be a doubling metric measure space. Let  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ . Then*

$$N_{p,q}^s(S) \subset \mathcal{B}_{p,q}^s(S) \subset \hat{\mathcal{B}}_{p,q}^s(S)$$

*and there is a constant  $C > 0$  such that*

$$\|\cdot\|_{\hat{\mathcal{B}}_{p,q}^s(S)} \leq \|\cdot\|_{\mathcal{B}_{p,q}^s(S)} \leq C \|\cdot\|_{N_{p,q}^s(S)},$$

*for each measurable set  $S \subset X$ .*

*If  $S$  satisfies measure density condition (1.1), then*

$$N_{p,q}^s(S) = \mathcal{B}_{p,q}^s(S) = \hat{\mathcal{B}}_{p,q}^s(S)$$

*with equivalent norms.*

**Theorem 7.5.** *Let  $X$  be a  $Q$ -regular, geodesic metric measure space. Let  $0 < s < 1$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ . Then  $\Omega \subset X$  is an extension domain for  $\mathcal{B}_{p,q}^s$  (resp. for  $\hat{\mathcal{B}}_{p,q}^s$ ) if and only if it satisfies measure density condition (1.1).*

**7.2. Triebel–Lizorkin spaces on subsets of  $\mathbb{R}^n$ .** Let  $S \subset \mathbb{R}^n$  be a measurable set. Let  $0 < s < 1$ ,  $0 < p, q < \infty$  and  $0 < r < \min\{p, q\}$ .

The Triebel–Lizorkin space  $\mathcal{F}_{p,q}^s(S)$  consists of functions  $u \in L^p(S)$ , for which the norm

$$\|u\|_{\mathcal{F}_{p,q}^s(S)} = \|u\|_{L^p(S)} + \|g\|_{L^p(S)},$$

where

$$g(x) = \left( \int_0^1 \left( t^{-s} \left( \frac{1}{|B(x,t)|} \int_{B(x,t) \cap S} |u(y) - u(x)|^r dy \right)^{1/r} \right)^q \frac{dt}{t} \right)^{1/q},$$

is finite. For  $S = \mathbb{R}^n$ , this definition coincides with the classical difference definition.

The Triebel–Lizorkin space  $\hat{\mathcal{F}}_{p,q}^s(S)$  consists of functions  $u \in L^p(S)$ , for which the norm

$$\|u\|_{\hat{\mathcal{F}}_{p,q}^s(S)} = \|u\|_{L^p(S)} + \|\hat{g}\|_{L^p(S)},$$

where

$$\hat{g}(x) = \left( \int_0^1 \left( t^{-s} \left( \frac{1}{|B(x,t)|} \inf_{c \in \mathbb{R}} \int_{B(x,t) \cap S} |u(y) - c|^r dy \right)^{1/r} \right)^q \frac{dt}{t} \right)^{1/q},$$

is finite. This definition, with  $r = 1$ ,  $p, q > 1$ , was used in [39].

**Remark 7.6.** If in the definitions above we integrate over  $(0, \infty)$  instead of  $(0, 1)$ , we end up with the equivalent norms.

Corresponding to Theorem 7.1 for Besov spaces, we have the following result.

**Theorem 7.7.** *Let  $0 < s < 1$ ,  $0 < p, q < \infty$ . Then*

$$(7.6) \quad M_{p,q}^s(S) \subset \mathcal{F}_{p,q}^s(S) \subset \hat{\mathcal{F}}_{p,q}^s(S)$$

and there is a constant  $C > 0$  such that

$$(7.7) \quad \|\cdot\|_{\hat{\mathcal{F}}_{p,q}^s(S)} \leq \|\cdot\|_{\mathcal{F}_{p,q}^s(S)} \leq C \|\cdot\|_{M_{p,q}^s(S)},$$

for each measurable set  $S \subset \mathbb{R}^n$ .

If  $S$  satisfies measure density condition (1.1), then

$$(7.8) \quad M_{p,q}^s(S) = \mathcal{F}_{p,q}^s(S) = \hat{\mathcal{F}}_{p,q}^s(S)$$

with equivalent norms.

*Proof.* Let  $S \subset \mathbb{R}^n$  be a measurable set. The first inequality in (7.7) is obvious because  $\hat{g}(x) \leq g(x)$ . To prove the second inequality, let  $u \in M_{p,q}^s(S)$ ,  $(g_k)_{k \in \mathbb{Z}} \in \mathbb{D}^s(u)$  and  $k \in \mathbb{Z}$ . Then, for almost every  $x \in S$ ,

$$\begin{aligned} & \frac{1}{|B(x, 2^{-k})|} \int_{B(x, 2^{-k}) \cap S} |u(x) - u(y)|^r dy \\ &= \sum_{j \geq k} \frac{1}{|B(x, 2^{-k})|} \int_{(B(x, 2^{-j}) \setminus B(x, 2^{-j-1})) \cap S} |u(x) - u(y)|^r dy \\ &\leq C \sum_{j \geq k} 2^{-jsr} \left( g_j(x)^r + \int_{B(x, 2^{-j})} (g_j(y)^r \chi_S(y)) dy \right) \\ &\leq C \sum_{j \geq k} 2^{-jsr} \mathcal{M}(g_j^r \chi_S)(x), \end{aligned}$$

and, hence,

$$\begin{aligned} g(x)^q &= \int_0^1 \left( t^{-s} \left( \frac{1}{|B(x, t)|} \int_{B(x, t) \cap S} |u(x) - u(y)|^r dy \right)^{1/r} \right)^q \frac{dt}{t} \\ &\leq C \sum_{k \geq 0} \left( 2^{ks} \left( \frac{1}{|B(x, 2^{-k})|} \int_{B(x, 2^{-k}) \cap S} |u(x) - u(y)|^r dy \right)^{1/r} \right)^q \\ &\leq C \sum_{k \geq 0} \left( \sum_{j \geq k} 2^{(k-j)sr} \mathcal{M}(g_j^r \chi_S)(x) \right)^{q/r} \\ &\leq C \sum_{j \in \mathbb{Z}} \left( \mathcal{M}(g_j^r \chi_S)(x) \right)^{q/r}, \end{aligned}$$

where the last inequality follows from Lemma 3.1. By the Fefferman–Stein vector valued maximal function theorem, we obtain

$$\begin{aligned} \|g\|_{L^p(S)} &\leq C \|(\mathcal{M}(g_k^r \chi_S))_{k \in \mathbb{Z}}\|_{L^{p/r}(\mathbb{R}^n, l^{q/r})}^{1/r} \leq C \|(g_k^r \chi_S)_{k \in \mathbb{Z}}\|_{L^{p/r}(\mathbb{R}^n, l^{q/r})}^{1/r} \\ &= C \|(g_k)_{k \in \mathbb{Z}}\|_{L^p(S, l^q)}. \end{aligned}$$

The claim follows by taking the infimum over all  $(g_k)_{k \in \mathbb{Z}} \in \mathbb{D}^s(u)$ .

If  $S$  satisfies measure density condition (1.1), then (7.8) follows from the proof of [11, Thm 3.1].  $\square$

By combining Theorems 7.7, 6.1 and 1.2, we obtain the second main result of this section.

**Theorem 7.8.** *Let  $0 < s < 1$ ,  $0 < p, q < \infty$ . Then  $\Omega \subset \mathbb{R}^n$  is an extension domain for  $\mathcal{F}_{p,q}^s$  (resp. for  $\hat{\mathcal{F}}_{p,q}^s$ ) if and only if it satisfies measure density condition (1.1).*

**Remark 7.9.** As in the case of Besov spaces, the definitions of Triebel–Lizorkin spaces and the results above have counterparts in metric setting.

**Remark 7.10.** For domains  $\Omega \subset \mathbb{R}^n$ , Triebel–Lizorkin spaces  $C_{p,q}^s(\Omega)$  consisting of functions  $u \in L^p(\Omega)$ , for which the norm

$$\|u\|_{C_{p,q}^s(\Omega)} = \|u\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)},$$

where

$$(7.9) \quad h(x) = \left( \int_0^{\tau\delta(x)} \left( t^{-s} \left( \inf_{c \in \mathbb{R}} \int_{B(x,t)} |u(y) - c|^r dy \right)^{1/r} \right)^q \frac{dt}{t} \right)^{1/q},$$

$0 < \tau < 1$  and  $\delta(x) = d(x, \Omega^c)$ , is finite have been studied in [36] and [30]. Since

$$M_{p,q}^s(\Omega) \subset \hat{\mathcal{F}}_{p,q}^s(\Omega) \subset C_{p,q}^s(\Omega),$$

Theorem 6.1 implies that  $C_{p,q}^s$ -extension domains are regular. The converse is not true. For example, the slit disc  $\Omega = B(0, 1) \setminus ([0, 1] \times \{0\}) \subset \mathbb{R}^2$  is regular, but it is clearly not a  $C_{p,q}^s$ -extension domain.

On the other hand, if  $\Omega \subset \mathbb{R}^n$  is an  $(\varepsilon, \delta)$ -domain, then  $C_{p,q}^s(\Omega)$  coincides with the other Triebel–Lizorkin spaces considered in this section. This follows, for example, from the extension results obtained in [36] and [30], and the characterization of extension domains for the spaces  $C_{p,q}^s$  below.

**Theorem 7.11.** *Let  $0 < s < 1$ ,  $0 < p, q < \infty$ . Then  $\Omega \subset \mathbb{R}^n$  is an extension domain for  $C_{p,q}^s$  if and only if it satisfies measure density condition (1.1) and  $C_{p,q}^s(\Omega) = M_{p,q}^s(\Omega)$ .*

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